Mathematical Tables

and other

Aids to Computation

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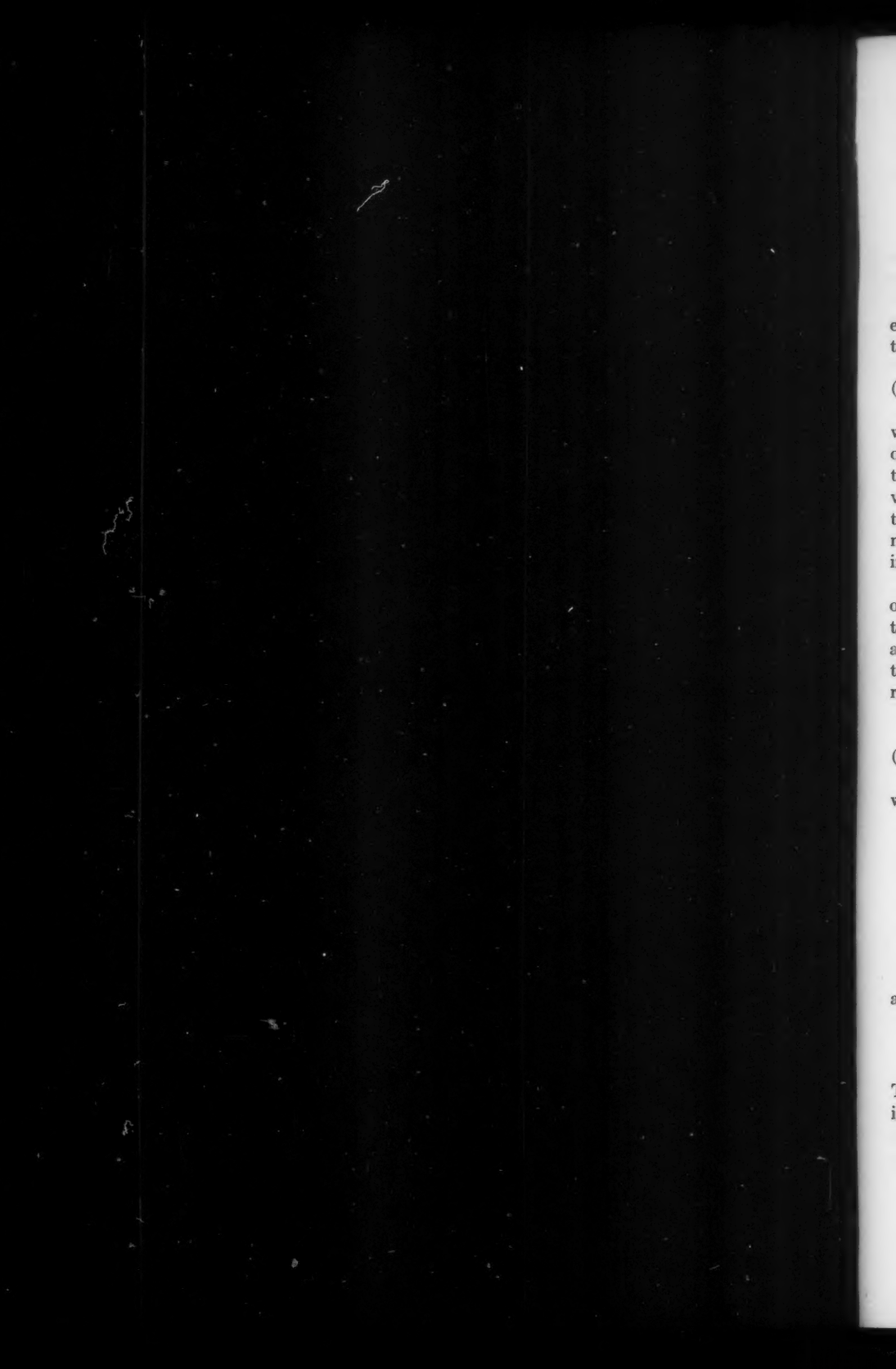
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A Method for the Numerical Integration of Ordinary Differential Equations

by L. Stoller and D. Morrison

1. Introduction. We consider the problem of solving numerically a differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$. Our point of departure is the formula

$$(1) \qquad y(x_0 + h) = y(x_0) + \int_{x_0}^{x_0+h} f[x, y(x)] \, dx,$$

where $y(x)$ denotes the solution of the differential equation. The idea is to use a quadrature formula to estimate the integral of (1). This requires knowledge of the integrand at specified arguments x_i in $(x_0, x_0 + h)$—hence we require the values of $y(x)$ at these arguments. A numerical integration method may be used to estimate $y(x)$ for the required arguments. In this way a numerical integration method is combined with a quadrature formula to obtain another numerical integration method.

A large number of methods may be devised, depending on which combination of quadrature formula and integration method is used. In particular, the Gauss two-point quadrature formula combined with the Runge-Kutta fourth order method appears to give excellent results [1]. We propose here the combination of the Radau three-point quadrature formula with the Runge-Kutta fourth order method. The resulting method seems to give greater accuracy with the same amount of work.

2. The Method. The Radau quadrature formula [2] gives

$$(2) \qquad y(x_0 + h) = y(x_0) + \frac{h}{2} [W_0 y'(x_0) + W_q y'(x_q) + W_p y'(x_p)] - \epsilon_Q,$$

where

$$W_0 = \frac{2}{9}$$

$$W_q = \left(\frac{8}{9} + \frac{\sqrt{6}}{18}\right) \qquad x_q = x_0 + h\left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right) = x_0 + hq$$

$$W_p = \left(\frac{8}{9} - \frac{\sqrt{6}}{18}\right) \qquad x_p = x_0 + h\left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right) = x_0 + hp$$

and

$$\epsilon_Q = \frac{-h^6}{72,000} \frac{d^5 y(\eta)}{dx^5}, \qquad x_0 < \eta < x_0 + h.$$

The method proceeds as follows: We use the Runge-Kutta fourth order method to integrate from x_0 to x_q and then from x_q to x_p, thus obtaining estimates y_q and y_p

for $y(x_q)$ and $y(x_p)$. Then formula (2) is used to obtain an estimate y_h of $y(x_0 + h)$:

$$(3) \qquad y_h = y(x_0) + \frac{h}{2} [W_0 f(x_0, y_0) + W_q f(x_q, y_q) + W_p f(x_p, y_p)].$$

3. The Error. We shall be concerned in this section with a description of the local truncation error of the method described in section 2. The local truncation error of a single step numerical integration method is defined as follows. Let $y(x)$ denote an exact solution of the equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$. Let y_h denote the approximation to $y(x_0 + h)$ obtained by the use of one step of the numerical method. Then the local truncation error $R(x_0, y_0, h)$ is defined by

$$(4) \qquad R(x_0, y_0, h) = y_h - y(x_0 + h).$$

The local truncation error of the Runge-Kutta method will be denoted by $R_1(x_0, y_0, h)$, and may be written in the form:

$$(5) \qquad R_1(x_0, y_0, h) = \phi(x_0, y_0)h^5 + 0(h^6).$$

If $y_1(x)$ denotes the exact solution of the differential equation which satisfies the initial condition $y_1(x_q) = y_q$ then the errors at x_q and x_p may be written

$$(6) \qquad y_q - y(x_q) = \phi(x_0, y_0)(qh)^5 + 0(h^6)$$

$$(7) \qquad y_p - y_1(x_p) = \phi(x_q, y_q)(p - q)^5 h^5 + 0(h^6).$$

Assuming the function ϕ is sufficiently smooth, we may replace $\phi(x_q, y_q)$ in (7) by $\phi = \phi(x_0, y_0)$ since this only affects the $0(h^6)$ term. The function $\eta(x) = y_1(x) - y(x)$ satisfies the differential equation

$$(8) \qquad \eta'(x) = f(x, y_1) - f(x, y).$$

An application of the mean value theorem gives

$$(9) \qquad \eta'(x) = f_y(x, \xi)\eta(x)$$

where $\xi = \xi(x)$ is a number between $y_1(x)$ and $y(x)$. It follows that

$$(10) \qquad \begin{aligned} \eta(x_p) &= \eta(x_q) + \eta'(x_q)(x_p - x_q) + \cdots \\ &= \eta(x_q)(1 + 0(h)). \end{aligned}$$

But by (6) $\eta(x_q) = \phi(qh)^5 + 0(h^6)$. Using the definition of $\eta(x)$ we have:

$$(11) \qquad y_1(x_p) - y(x_p) = \phi(qh)^5 + 0(h^6).$$

Now, combining (7) and (11)

$$(12) \qquad y_p - y(x_p) = \phi(qh)^5 + \phi(p - q)^5 h^5 + 0(h^6).$$

The error in the derivatives may now be computed:

$$(13) \qquad f(x_q, y_q) - f(x_q, y(x_q)) = f_y\phi(qh)^5 + 0(h^6)$$

$$(14) \qquad f(x_p, y_p) - f(x_p, y(x_p)) = f_y[\phi(qh)^5 + \phi(p - q)^5 h^5] + 0(h^6)$$

where we have again used the mean value theorem. We have also assumed that $f_y(x, y)$ is constant and put the resulting error into the $0(h^6)$ term. We may now

TABLE I

	Gauss	Radau
Number of evaluations of $f(x, y)$ per step	9	9
Order of local truncation error	h^5	h^6
ϵ_Q (error due to quadrature)	$-\dfrac{h^5 y^v(\eta)}{4120}$	$-\dfrac{h^6 y^{vi}(\eta)}{72,000}$
ϵ_I (error due to integration)	$\dfrac{11-5\sqrt{3}}{72} h^6 f_y \phi$	$\dfrac{24-9\sqrt{6}}{125} h^6 f_y \phi$

TABLE II

Problem	Method	Evaluations of $f(x, y)$	Error at $x = 1$
(a)	Gauss	36	2.74×10^{-6}
	Radau	36	0.298×10^{-6}
	Runge-Kutta	64	0.298×10^{-6}
(b)	Gauss	144	0.229×10^{-4}
	Radau	144	0.0977×10^{-4}
(c)	Runge-Kutta	128	0.815×10^{-3}
	Gauss	126	0.282×10^{-3}
	Radau	126	0.116×10^{-3}

compute the local truncation error R of the method of section 2. From (2), (3), (13) and (14) we obtain

$$R(x_0, y_0, h) = y_4 - y(x_0 + h)$$

(15)
$$= \frac{h}{2} \{ W_q f_y \phi (qh)^5 + W_p f_y \phi [(qh)^5 + (p - q)^5 h^5] \} + \epsilon_Q + 0(h^6)$$

$$= \epsilon_I + \epsilon_Q + 0(h^6).$$

A similar analysis may be performed on the method which combines Gauss quadrature with Runge-Kutta integration, and Table I summarizes the results. Table II gives some of the numerical results which were obtained on the 1103A. Also included in Table II is a comparison with results obtained from the use of the Runge-Kutta formula alone.

Problems

(a) $f(x, y) = y, x_0 = 0, y_0 = 1.$ Solution: $y = e^x$

(b) $f(x, y) = \dfrac{5y}{1 + x}, x_0 = 0, y_0 = 1.$ Solution: $y = (1 + x)^5$

(c) $f(x, y) = \dfrac{6y}{1 + x}, x_0 = 0, y_0 = 1.$ Solution: $y = (1 + x)^6$

Space Technology Laboraties, Inc.
Los Angeles, California

1. P. HENRICI, Lecture notes on the numerical solution of ordinary differential equations (UCLA).

2. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.
3. H. FLATT & S. CONTE, *Integration of Ordinary Differential Equations*, NN-21, August 30, 1956, The Ramo-Wooldridge Corporation, Los Angeles, California.

Numerical Evaluation of Multiple Integrals II

by Preston C. Hammer and Arthur H. Stroud

1. Introduction. In the first paper of this title [1], Hammer and Wymore introduced methods whereby integration formulas of the form

$$(1) \qquad \int_R w(x)f(x) \, dx \doteq \sum a_i f(\xi_i)$$

which are known for special regions (in n-dimensional euclidean space E_n) may be used to determine formulas for other regions. They also showed, in some cases, how the symmetry of a region may simplify the task of finding integration formulas for the region.

To facilitate numerical integration over regions in higher dimensional spaces, we summarize the most important formulas (1) and review methods by which formulas for classes of regions may be obtained from them. These methods enable one to obtain formulas for regions which we consider too special to warrant particular formulas.

2. Regions with symmetry. In deriving numerical integration formulas it is possible to obtain explicit formulas with comparatively little effort when the region and the formula both are assumed to have certain kinds of symmetry. Formulas precise for polynomial functions involving minimal numbers of points can be derived, in principle, by solving simultaneous algebraic equations by general elimination procedures leading to polynomials which have as roots the solutions of the system. However, the manipulative work in achieving such solutions is forbidding in magnitude and can probably be done effectively only with high speed computers. Moreover, the solutions may turn out to be complex numbers and the determinations of approximate values for the solutions will involve the numerical solution of high degree polynomial equations. For example, the general seventh degree polynomial in three variables has 120 terms so that the determination of a numerical integration formula precise for all such polynomials over an arbitrary region R would lead to the problem of solving a system of 120 equations of algebraic (non-linear) character. In [1] it is shown that for certain symmetrical regions with a symmetrical formula the problem is reduced to seven simultaneous algebraic equations for which explicit solutions are easily derived. The problem of finding integrals of monomials to establish the equations to be solved also will be a difficult problem for many regions.

A set S in E_n is said to be *fully symmetrical* provided $x \, \epsilon \, S$ implies that every

point y obtainable from x by permutations and/or changes of sign of the coordinates of x is also in S. A function g defined on a fully symmetrical set S is said to be *fully symmetrical* provided $g(x) = g(y)$ whenever y may be obtained from x as above. A numerical integration formula of the form (1) is said to be *fully symmetrical* if the set of all evaluation points ξ_1, ξ_2, $\cdots$ forms a fully symmetrical set and the weight function $a_j = a(\xi_j)$ is a fully symmetrical function. In [1] full symmetry was indicated simply as symmetry. Full symmetry is defined here in reference to the coordinate system, but could readily be extended to include all sets which result from rigid motions of those included in the definition. For use in this paper the definition given is adequate.

Now let $w(x)$ be a fully symmetrical function on a fully symmetrical region R. If a fully symmetrical integration formula $\sum a_j f(\xi_j)$ is equal to $\int_R f(x)\, dx$ for all polynomials of at most degree k then, in principle, a fully symmetrical formula $\sum b_j f(\nu_j)$ may be found which gives the value of $\int_R w(x) f(x)\, dx$ for the same class of polynomials using the same algebraic manipulations as in finding a_j and ξ_j. That is, the algebraic form in which a_j and ξ_j appear is the same form as that in which b_j and ν_j appear. Since the existence of solutions of these equations may depend on the values of the constants it cannot be stated that b_j and ν_j can always be found if a_j and ξ_j were determined. This remark is important since for spherical regions weight functions $w(r)$ depending only on $r = |x|$ are common and such weight functions are fully symmetrical.

We anticipate that for many useful weight functions $w(r)$ there will be numerical integration formulas over the spherical shell involving the same number of points as formulas for $w = 1$.

3. Summary of numerical integration formulas. We summarize known formulas of the form (1) with $w(x) = 1$ for regions in euclidean spaces of dimension ≥ 2. We give formulas for cubes, spheres, and simplexes.

Tables 1 and 2 give certain fully symmetrical formulas for cubes and spheres respectively. The points in these formulas are divided into one or more fully symmetric subsets; each subset is generated by any one of its points (a generator). Below are generators for each fully symmetric subset occurring in the formulas of these tables. With the generators we give the number of points in the subset and the weight for each point in the subset.

Generator	Number of points	Weight
$(0,\ \ 0,\ \ 0,\ \ 0,\ \cdots, 0)$	1	a_0
$(\nu,\ \ 0,\ \ 0,\ \ 0,\ \cdots, 0)$	$2n$	a_1
$(\xi_1,\ \xi_1, 0,\ \ 0,\ \cdots, 0)$	$2n(n-1)$	b_1
$(\xi_2,\ \xi_2, 0,\ \ 0,\ \cdots, 0)$	$2n(n-1)$	b_2
$(\eta_1,\ \eta_1,\ \eta_1, 0,\ \cdots, 0)$	$\frac{4}{3}n(n-1)(n-2)$	c_1
$(\eta_2,\ \eta_2,\ \eta_2, 0,\ \cdots, 0)$	$\frac{4}{3}n(n-1)(n-2)$	c_2

With these generators we can obtain formulas of degree ≤ 7. (A formula is of degree k if it is exact for polynomials of at degree no greater than k.) The system of

TABLE 1. *Formulas for cubes*

1-2	2-cube	degree 3	4 points
	$\nu = 0.8164965809277260$		$a_1 = 1.000000000000000$
2-2	2-cube	degree 5	9 points
			$a_0 = 0.7901234567901235$
	$\nu = 0.7745966692414834$		$a_1 = 0.4938271604938272$
	$\xi_1 = 0.7745966692414834$		$b_1 = 0.3086419753086420$
3-2	2-cube	degree 7	12 points
	$\nu = 0.9258200997725515$		$a_1 = 0.2419753086419753$
	$\xi_1 = 0.3805544332083157$		$b_1 = 0.5205929166673945$
	$\xi_2 = 0.8059797829185987$		$b_2 = 0.2374317746906302$
1-3	3-cube	degree 3	6 points
	$\nu = 1.000000000000000$		$a_1 = 1.3333333333333333$
2-3	3-cube	degree 5	19 points
			$a_0 = 2.0740740740740741$
	$\nu = 0.7745966692414834$		$-a_1 = 0.2469135802469136$
	$\xi_1 = 0.7745966692414834$		$b_1 = 0.6172839506172840$
4-3	3-cube	degree 5	14 points
	$\nu = 0.7958224257542215$		$a_1 = 0.8864265927977839$
	$\eta_1 = 0.7587869106393281$		$c_1 = 0.3351800554016621$
5-3	3-cube	degree 7	27 points
			$a_0 = 0.7880734827442106$
			(0.9478945552646438)
	$\nu = 0.8484180114722525$		$a_1 = 0.4993690023077203$
	(1.2795818594182734)		(0.0424299394912215)
	$\xi_1 = 1.1064128986267175$		$b_1 = 0.0323037423340374$
	(0.7000972875523367)		(0.5032755687554778)
	$\eta_1 = 0.6528164721016912$		$c_1 = 0.4785084494251273$
	(0.8550442581681327)		(0.0947773728402868)
6-3	3-cube	degree 7	34 points
	$\nu = 0.9258200997725515$		$a_1 = 0.2957475994513032$
	$\xi_1 = 0.9258200997725515$		$b_1 = 0.0941015089163237$
	$\eta_1 = 0.7341125287521153$		$c_1 = 0.2247031747656014$
	$\eta_2 = 0.4067031864267161$		$c_2 = 0.4123338622714356$
1-n	n-cube	degree 3	$2n$ points
	$\nu = \sqrt{\dfrac{n}{3}}$		$a_1 = \dfrac{2^{n-1}}{n}$
2-n	n-cube	degree 5	$2n^2 + 1$ points
			$a_0 = \dfrac{2^{n-1}}{81}(25n^2 - 115n + 162)$
	$\nu = \sqrt{\dfrac{3}{5}}$		$a_1 = \dfrac{2^{n-1}}{81}(70 - 25n)$
	$\xi_1 = \sqrt{\dfrac{3}{5}}$		$b_1 = \dfrac{(25)2^{n-1}}{162}$

TABLE 2. *Formulas for spheres*

11-2	2-sphere	degree 3	4 points

$$\nu = 0.7071067811835475 \qquad a_1 = 0.7853981633974483$$

12-2	2-sphere	degree 5	9 points

$$a_0 = 0.5235987755982989$$
$$\nu = 0.7071067811865475 \qquad a_1 = 0.5235987755982989$$
$$\xi_1 = 0.7071067811865475 \qquad b_1 = 0.1308996938995747$$

13-2	2-sphere	degree 7	12 points

$$\nu = 0.8660254037844386 \qquad a_1 = 0.2327105669325773$$
$$\xi_1 = 0.3229149920674005 \qquad b_1 = 0.3870777960062264$$
$$\xi_2 = 0.6441713103894646 \qquad b_2 = 0.1656098004586446$$

11-3	3-sphere	degree 3	6 points

$$\nu = 0.7745966692414834 \qquad a_1 = 0.6981317007977318$$

12-3	3-sphere	degree 5	19 points

$$a_0 = 0.2792526803190927$$
$$\nu = 0.6546536707079771 \qquad a_1 = 0.3257947937056082$$
$$\xi_1 = 0.6546536707079771 \qquad b_1 = 0.1628973968528041$$

14-3	3-sphere	degree 5	14 points

$$\nu = 0.6822591268536840 \qquad a_1 = 0.5523611797267854$$
$$(1.2387584445019331) \qquad (0.5082460976245486)$$
$$\eta_1 = 0.6082048823194740 \qquad c_1 = 0.1093278908032098$$
$$(0.4189765704395655) \qquad (0.4854803182764577)$$

15-3	3-sphere	degree 7	27 points

$$a_0 = 0.4156003482691997$$
$$(0.4441396821009518)$$
$$\nu = 0.8326956271382924 \qquad a_1 = 0.1994483077968051$$
$$(0.9410448241002225) \qquad (0.0957384071760634)$$
$$\xi_1 = 0.7476506947169606 \qquad b_1 = 0.0380676101171267$$
$$(0.5460414781242386) \qquad (0.2508385364520637)$$
$$\eta_1 = 0.4294549987784796 \qquad c_1 = 0.2649610860413550$$
$$(0.6604983415547611) \qquad (0.0200197052755367)$$

11-n	n-sphere	degree 3	$2n$ points

$$\nu = \sqrt{\frac{n}{n+2}} \qquad\qquad a_1 = \frac{1}{2n} I(1)^*$$

12-n	n-sphere	degree 5	$2n^2 + 1$ points

$$a_0 = \frac{n^3 - 3n^2 - 10n + 36}{18n + 36} I(1)$$

$$\nu = \sqrt{\frac{3}{n+4}} \qquad -a_1 = \frac{n^2 - 16}{18n + 36} I(1)$$

$$\xi_1 = \sqrt{\frac{3}{n+4}} \qquad b_1 = \frac{n+4}{36n + 72} I(1)$$

$$^* I(1) = \pi^{n/2} / \Gamma\left(\frac{n}{2} + 1\right)$$

TABLE 3. *Formulas for the circle $x_1^2 + x_2^2 \leq 1$*

Points		Weights
17.	**degree 5**	**7 points**
(0.0000000000000000,	0.0000000000000000)	0.7853981633974483
(±0.8164965809277260,	0.0000000000000000)	0.3926990816987242
(±0.4082482904638630,	±0.7071067811865475)	0.3926990816987242
18.	**degree 7**	**16 points**
(±0.4247082002778669,	±0.1759198966061612)	0.1963495408493621
(±0.1759198966061612,	±0.4247082002778669)	0.1963495408493621
(±0.8204732385702833,	±0.3398511429799874)	0.1963495408493621
(±0.3398511429799874,	±0.8204732385702833)	0.1963495408493621
19.	**degree 9**	**21 points**
(0.0000000000000000,	0.0000000000000000)	0.3490658503988659
(±0.5505043204538557,	±0.2280263556769715)	0.2012527133278051
(±0.2280263556769715,	±0.5505043204538557)	0.2012527133278051
(±0.9192110607898046,	0.0000000000000000)	0.1012918735702551
(0.0000000000000000,	±0.9192110607898046)	0.1012918735702551
(±0.7932084745126058,	±0.4645097310495256)	0.0971672002859332
(±0.4645097310495256,	±0.7932084745126058)	0.0971672002859332
20.	**degree 11**	**32 points**
(±0.3357106870197288,	0.0000000000000000)	0.1090830782496456
(0.0000000000000000,	±0.3357106870197288)	0.1090830782496456
(±0.2373833033084449,	±0.2373833033084449)	0.1090830782496456
(±0.7071067811865475,	0.0000000000000000)	0.1161047224304262
(0.0000000000000000,	±0.7071067811865475)	0.1161047224304262
(±0.6125369400823741,	±0.3532683074300921)	0.1164805639842198
(±0.3532683074300921,	±0.6125369400823741)	0.1164805639842198
(±0.8157480497746617,	±0.4710132205252606)	0.0727157433213629
(±0.4710132205252606,	±0.8157480497746617)	0.0727157433213629
(±0.9419651451198933,	0.0000000000000000)	0.0727346698565653
(0.0000000000000000,	±0.9419651451198933)	0.0727346698565653
21.	**degree 15**	**64 points**
(±0.2584361661674054,	±0.0514061496288813)	0.0341505695624825
(±0.0514061496288813,	±0.2584361661674054)	0.0341505695624825
(±0.5634263397544869,	±0.1120724670846205)	0.0640242008621985
(±0.1120724670846205,	±0.5634263397544869)	0.0640242008621985
(±0.4776497869993547,	±0.3191553840796721)	0.0640242008621985
(±0.3191553840796721,	±0.4776497869993547)	0.0640242008621985
(±0.8028016728473508,	±0.1596871812824163)	0.0640242008621985
(±0.1596871812824163,	±0.8028016728473508)	0.0640242008621985
(±0.6805823955716280,	±0.4547506180649039)	0.0640242008621985
(±0.4547506180649039,	±0.6805823955716280)	0.0640242008621985
(±0.2190916025980981,	±0.1463923286035535)	0.0341505695624825
(±0.1463923286035535,	±0.2190916025980981)	0.0341505695624825
(±0.9461239423417719,	±0.1881957532057769)	0.0341505695624825
(±0.1881957532057769,	±0.9461239423417719)	0.0341505695624825
(±0.8020851487551318,	±0.5359361621905023)	0.0341505695624825
(±0.5359361621905023,	±0.8020851487551318)	0.0341505695624825

TABLE 4. *Formulas for the triangle and tetrahedron*

Triangle, degree 2

$$\left(\frac{1}{6}, \frac{1}{6}\right) \frac{1}{6}$$

$$\left(\frac{4}{6}, \frac{1}{6}\right) \frac{1}{6}$$

$$\left(\frac{1}{6}, \frac{4}{6}\right) \frac{1}{6}$$

Triangle, degree 3

$$\left(\frac{1}{5}, \frac{1}{5}\right) \frac{25}{96}$$

$$\left(\frac{3}{5}, \frac{1}{5}\right) \frac{25}{96}$$

$$\left(\frac{1}{5}, \frac{3}{5}\right) \frac{25}{96}$$

$$\left(\frac{1}{3}, \frac{1}{3}\right) \frac{-27}{96}$$

Tetrahedron, degree 2

$$\left(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}\right) \frac{1}{24}$$

$$\left(\frac{5+3\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}\right) \frac{1}{24}$$

$$\left(\frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}\right) \frac{1}{24}$$

$$\left(\frac{5-\sqrt{5}}{20}, \frac{5-\sqrt{5}}{20}, \frac{5+3\sqrt{5}}{20}\right) \frac{1}{24}$$

Tetrahedron, degree 3

$$\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \frac{9}{120}$$

$$\left(\frac{3}{6}, \frac{1}{6}, \frac{1}{6}\right) \frac{9}{120}$$

$$\left(\frac{1}{6}, \frac{3}{6}, \frac{1}{6}\right) \frac{9}{120}$$

$$\left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}\right) \frac{9}{120}$$

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \frac{-16}{120}$$

equations which a formula of either Table 1 or 2 satisfies can be obtained from the following system by making suitable omissions and substitutions. We use $I(f(x)) = \int_R f(x)\, dx$.

$$a_0 + 2na_1 + 2n(n-1)(b_1 + b_2) + \tfrac{4}{3}n(n-1)(n-2)(c_1 + c_2) = I(1)$$

$$2a_1\nu^2 + 4(n-1)(b_1\xi_1^2 + b_2\xi_2^2) + 4(n-1)(n-2)(c_1\eta_1^2 + c_2\eta_2^2) = I(x_1^2)$$

$$2a_1\nu^4 + 4(n-1)(b_1\xi_1^4 + b_2\xi_2^4) + 4(n-1)(n-2)(c_1\eta_1^4 + c_2\eta_2^4) = I(x_1^4)$$

$$4(b_1\xi_1^4 + b_2\xi_2^4) + 8(n-2)(c_1\eta_1^4 + c_2\eta_2^4) = I(x_1^2 x_2^2)$$

$$2a_1\nu^6 + 4(n-1)(b_1\xi_1^6 + b_2\xi_2^6) + 4(n-1)(n-2)(c_1\eta_1^6 + c_2\eta_2^6) = I(x_1^6)$$

$$4(b_1\xi_1^6 + b_2\xi_2^6) + 8(n-2)(c_1\eta_1^6 + c_2\eta_2^6) = I(x_1^4 x_2^2)$$

$$8(c_1\eta_1^6 + c_2\eta_2^6) = I(x_1^2 x_2^2 x_3^2)$$

Tables 1 and 2 give formulas for the cube with vertices $(\pm1, \pm1, \cdots, \pm1)$ and for the sphere of radius 1 with centroid at the origin. The formulas have been numbered in the form $h - k$ where h is the formula number and k is the dimension of the space.

Formula 1 (for the n-cube) was first given by Tyler [3]. For $n > 3$ this formula has the disadvantage of having the points exterior to the cube. Stroud [4] has given a comparable formula with points interior to the cube for all n. Formula 3–2 was also given by Tyler and the two formulas given as formula 5–3 were given by Clerk-Maxwell (p. 66[1]). Formula 6–3 though containing more points may be more desirable than formulas 5–3, which have points exterior to the 3-cube.

Peirce has obtained formulas of arbitrarily high degree for both the circular annulus [5] and the spherical shell [6], all the points being interior to the regions. Table 3 gives certain of his formulas for the circle of unit radius with center at the origin. Formulas 17, 18, and 21 are members of the class of arbitrarily high degree. Formulas 19 and 20 are not of this class.

Hammer and Stroud [7] have given formulas of degrees 2 and 3 consisting of $n + 1$ and $n + 2$ points respectively for the n-simplex. Table 4 gives these formulas for the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

4. Extensions of formulas. Because a formula for a particular region may not always be available, it is desirable to use the known formulas to as large an extent as possible. In this discussion we limit ourselves to formulas of the form (1). In general, however, we assume numerical integration includes every manner of obtaining estimates including contour integration, expansion of the integrand, evaluation in finite terms and so on.

We give the following five methods for extension of formulas:

1. Transformation from a region R to a region S when a formula for R is known. This is discussed in I. This method is limited in practice by the difficulty of finding transformations.

2. If the region is a cartesian product of two or more regions of lower dimension and formulas are known for all or some of the factor regions the work of determining a formula is reduced as shown in I. For example, with formulas for the line segment and Peirce's formulas for the circular annulus formulas for truncated cylindrical shells may be determined.

3. If a region is a finite cone based on a region for which a formula is given a formula may be determined for the cone using the theory and tables given by Hammer, Marlowe and Stroud and by Fishman [8, 9].

4. Decompositions of regions into subregions for which formulas are known. For example, polygonal regions may be treated with the triangle formulas of Hammer, Marlowe, and Stroud. When feasible one may "subtract" to obtain estimates for integrals over regions representable as differences between regions for which formulas are available.

5. Approximate the region with a suitable region for which the integral may be approximated. Thus bounded planar regions may be approximated by polygonal regions.

Decompositions of a region into simplexes or cubes and the factoring of a region into cartesian factors both have practical limitations for higher dimensions. The number of simplexes of equal volume in the n-cube, for example, is $n!$. The cartesian product methods multiply the numbers of points so that it is not feasible to develop formulas for the hypercube by cartesian product methods if the dimension is great.

It is our impresson that triangulation will not be very useful except for 2 and 3 dimensions. Cartesian product methods may be useful up to, say, dimension 30 if one starts with efficient 3-dimensional factor formulas and uses a high-speed calculator.

With the above remarks in mind it is clear that the limitations on the classical type formulas are not as severe as some had thought and it appears as if the necessity for Monte-Carlo methods in dimensions of order 10 or less definitely will decrease. On the other hand the difficulty of error analysis will still mean that in the actual applications even the classical formulas will require empirical error estimates.

5. Concluding remarks. We have given here certain numerical integration formulas and indicated the extent of some not given. While the literature of the subject seems to be rather small, we cannot claim completeness in that regard. Bourget [10] discusses means of generating numerical integration formulas using orthogonal polynomials for the circular disk. While his method gives a number of points comparable to that of Peirce, the connection between the two has yet to be established. Since Peirce's results are more general (applicable to the annulus), more available, and his approach simpler, we have presented them.

Appell and Kampe de Feriet [11] discuss the use of orthogonal polynomials for numerical integration over the circular disk. Their applications are not as extensive as those of Bourget but the discussion is simpler. The thesis of Angelescu [12] we have yet to study. Thacher [13] gives methods of deriving formulas for the hypercube of degrees 2 and 3. His matrix methods provide an interesting point of view. The paper [14] of Lauffer contains formulas for the simplex based on a regular lattice of points in the simplex. The degree may be arbitrarily large.

Formulas with non-constant weight functions have not been derived. We have indicated that such formulas for symmetrical weight functions might be achieved but the presentation of such tables should hinge on the indicated usefulness of particular weight functions.

Finally, we are indebted to Miss Beverly Ferner and Mr. Richard Hetherington for carrying out or checking certain calculations. We are indebted to Dr. Z. Kopal for calling our attention to certain references of the literature.

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1. P. C. Hammer & A. W. Wymore, "Numerical evaluation of multiple integrals I," *MTAC*, v. 11, 1957, p. 59–67.
2. C. C. MacDuffee, *The Theory of Matrices*, Chelsea, New York, 1946, reprint.
3. G. W. Tyler, "Numerical integration of functions of several variables," *Canadian Jn. Math.*, v. 5, 1953, p. 393–412.
4. A. H. Stroud, "Remarks on the disposition of points in numerical integration formulas," *MTAC*, v. 11, 1957, p. 257–261.
5. W. H. Peirce, "Numerical integration over the planar annulus," Soc. Industrial and Applied Math., *Jn.*, v. 5, 1957, p. 66–73.
6. W. H. Peirce, "Numerical integration over the spherical shell," *MTAC*, v. 11, 1957, p. 244–249.
7. P. C. Hammer & A. H. Stroud, "Numerical integration over simplexes," *MTAC*, v. 10, 1956, p. 137–139.
8. P. C. Hammer, O. J. Marlowe & A. H. Stroud, "Numerical integration over simplexes and cones," *MTAC*, v. 10, 1956, p. 130–137.
9. H. Fishman, "Numerical integration constants," *MTAC*, v. 11, 1957, p. 1–9.
10. H. Bourget, "Sur une extension de la méthode de quadrature de Gauss," Hebdomadaires des séances de l'Académie des Sciences, *Comptes Rendus*, v. 126, 1898, p. 634–636.

11. P. Appell & J. Kampé de Fériet, *"Fonctions hypergéométriques et hypersphériques; Polynomes d'Hermite,"* Gauthier-Villars, Paris, 1926.

12. A. Angelescu, "Sur des polynomes generalisant les polynomes de Legendre et d'Hermite et sur le calcul approché des intégrales multiples," Thesis, Paris, 1916.

13. H. C. Thacher, Jr., "Optimum quadrature formulas in s dimensions," *MTAC*, v. 11, 1957, p. 189–194.

14. R. Lauffer, "Interpolation mehrfacher integrale," *Archiv. der Math.*, v. 6, 1955, p. 159–164.

Appendix A. *Integration formulas and interpolating polynomials.* In the one-dimensional case certain numerical integration formulas may be considered as giving the integral of an interpolating polynomial agreeing with the integrand at the points of evaluation. For example, this is true of the Gauss formulas and of Simpson's rule for one step of two subintervals. In higher dimensions formulas may not always be interpreted as integrals of interpolating polynomials agreeing with the integrand at the evaluation points. For example, one may obtain a numerical integration formula for the square which is exact for every polynomial of the form $\sum \sum c_{ij} x^i y^j$, $0 \leq i + j \leq 7$, and which has 36 evaluation points in a regular square lattice array. However, if this formula is applied to a non-polynomial integrand function, the approximate value usually cannot be interpreted as the integral of a seventh degree polynomial agreeing with the integrand at the 36 points. This is true since it is generally impossible to determine a seventh degree polynomial assuming arbitrarily specified values at 36 points of a square lattice as these points lie on a sixth degree algebraic curve. This situation is summarized in the following theorem.

Theorem 1. Let P be a linear space of functions (of n variables) which has m linearly independent functions as a basis. Let $x_1, \cdots, x_m$ be m points in E_n and let $c_1, \cdots, c_m$ be m arbitrary real numbers. A necessary and sufficient condition that there always exist a function $p \, \epsilon \, P$ such that $p(x_i) = c_i$ ($i = 1, \cdots m$) is that there exist no $f \, \epsilon \, P$ ($f \neq 0$) such that $f(x_i) = 0$ ($i = 1, \cdots, m$). That is, $x_1, \cdots, x_m$ must be independent with regard to P.

The example and theorem 1 indicate that certain types of error analysis depending on interpolation are not applicable for certain formulas in higher dimensions.

Cartesian product formulas, however, may be considered as integrals of interpolating polynomials provided the formulas from which they are derived can be considered in this manner. More precisely, let $L_1 : [f_1, \cdots, f_m]$ and $L_2 : [g_1, \cdots, g_n]$ be two linear function spaces with the linearly independent bases $f_1, \cdots, f_m$ and $g_1, \cdots, g_n$ respectively and let f_i be defined for all $x \, \epsilon \, R$ and g_k for all $y \, \epsilon \, S$. Further, let $x_1, \cdots, x_m$ in R and $y_1, \cdots, y_n$ in S be points such that the matrices $F = (f_i(x_j))$ and $G = (g_k(y_k))$ are non-singular. Then for each set of mn constants c_{jk} there exists a unique function $h \, \epsilon \, L_1 \times L_2$ such that $h(x_j, y_k) = c_{jk}$ ($j = 1, \cdots, m$; $k = 1, \cdots, n$). The proof of this follows from the relation $| F \times G | = F |^n | G |^m$ (see MacDuffee [2], p. 81).

TECHNICAL NOTES AND SHORT PAPERS

Some Computations of Wilson and Fermat Remainders

by Carl-Erik Fröberg

The Wilson remainders

$$W_p \equiv \frac{(p - 1)! + 1}{p} \ (\text{mod. } p)$$

have been computed by Goldberg (J. Lond. Math. Soc. 28, 109, 1953, p. 252) for all primes $p < 10{,}000$. He found 563 to be the third Wilson prime, the two others being 5 and 13. These computations have been continued on SMIL, the electronic computer of Lund University, for $10{,}000 < p < 30{,}000$, but no zeros were found in this interval.

The Fermat remainders

$$F_p \equiv \frac{2^{p-1} - 1}{p} \ (\text{mod. } p)$$

have been computed for all $p < 50{,}000$. Previously it was known that $F_p = 0$ for $p = 1093$, and now it is found that $p = 3511$ is the second solution.

A table of W_p and F_p will be published elsewhere.

Note, added in proof:

Professor D. H. Lehmer has kindly informed me that the solution $p = 3511$ was obtained by N. J. W. H. Beeger about 1930.

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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

126[C].—OLIVER L. I. BROWN, "A short table for computing seven-place logarithms," Marchant Calculators, Inc., Oakland, California, Table No. 107, 1958, 2 p., $8\frac{1}{2}''$ x $11''$. May be obtained free of charge by writing to Marchant Calculators, 6701 San Pablo Avenue, Oakland, California.

A table of $h + \log x$, $x = 1(.01)1.49(.02)2.15(.03)2.84(.04)3.60(.05)4.35(.06)$-5.07(.07)5.77(.08)6.49(.09)7.21(.1)8.01(.11)8.67(.12)9.87, 8D, where $0 < h \leqq 5 \times 10^{-8}$ this set of intervals and the chorus of h permit calculation of logarithms to 7D by one multiplication, one division and one addition using the formula $\log x = \log a + 0.868589(x - a)/(x + a)$, where a is the table argument nearest x. This formula is a truncated Taylor's series. Directions for use both direct and inverse are included.

The source of the logarithms is not noted. A few randomly chosen values were checked against [1] and no discrepancies noted.

Hastings [2] gets comparable accuracy using a five term polynomial of ninth degree in $(x - 1)/(x + 1)$ for $10^{-\frac{1}{3}} \leq x \leq 10^{\frac{1}{3}}$.

C. B. T.

1. SPENCELY, SPENCELEY & EPERSON, *Smithsonian Logarithmic Tables* to *Base* e *and Base* 10. Smithsonian Misc. Collections, v. 118, Washington, 1952, [MTAC, v. 6, 1952, Rev. 992, p. 150–151.]
2. CECIL HASTINGS, JR., *Approximations for Digital Computers*, Princeton, 1955, p. 128.

127[F].—F. V. TARBELL, "Table of least common denominators," 1 Sheet, $8\frac{1}{2}'' \times 11''$. Deposited in UMT File.

This is a table of the least common denominators of all the positive integers not exceeding n for n a power of a prime not exceeding 191. Since 193 is a prime, the effective top of the table is 192. For numbers which are not prime powers the function tabulated has the same value as for the next smaller prime power, so intervening entries are not made.

The table was prepared for use in constructing harmonic series. While it certainly must contribute, one should note that the largest entry has 83 decimal digits and is inconveniently large for computing machines as it stands.

No indication of accuracy checks is given.

C. B. T.

128[G, F].—JOHN RIORDAN, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958, x + 244 p., 23 cm. Price $8.50.

The reader who has had to live for so long with the three ancients in Combinatorial Analysis, Netto, Whitworth and MacMahon, owes very careful and sympathetic scrutiny to a new introduction in the field, and in Riordan's text he will find a book worthy of his attention. There will be some readers who will object to what has not been included, but what is there will be received with enthusiasm.

In his preface the author states "... the main emphasis throughout is on finding the *number of ways there are* of doing some well defined operation." He has very assiduously followed this policy and in so doing has omitted certain modern aspects of his subject which are of great interest to many readers. Latin rectangles are barely touched on, and block designs and finite planes are completely absent. In short, matters relating to existence and constructibility are missing.

As is to be expected from the author's personal interests the emphasis is on counting by means of generating functions, which are introduced as early as page seven. However, other methods are not neglected and well known results are usually proved in several ways with scholarly reference to their origin.

Chapter one is a survey of permutations and combinations. Chapter two introduces generating functions and their algebras as well as Stirling numbers. Chapter three is a short introduction to the principle of inclusion and exclusion. Chapter four is an introduction to the enumeration of permutations according to their cycle structure. The reader is made acquainted with the works of J. Touchard and references are furnished. Chapter five is a rapid survey of the theory of occupancy with reference to MacMahon. Chapters six, seven and eight are primarily the work

of the author and his collaborators and are about partitions, compositions, trees, networks and partitions with restricted position.

The subject of the text is very intimately associated with tables and the author introduces twenty-two of them in the body of his text. Their use is illustrated and in some instances he derives relationships for the functions tabulated. All of the tables are short. For example, the table for the Stirling numbers of the second kind, $S(n, k)$, covers the range $n = 0(1)10$, $k = 0(1)10$. It is unfortunate that references to more extensive tables were not included.

This book is very well suited for a classroom text on the material it covers and should prove useful as such to mathematicians, statisticians, numerical analysts, applied mathematicians and engineers. Each chapter terminates with a set of problems and there are 196 in all. These are well designed to extend the text as well as to increase the student's technique, and this is important in a subject where technique is more significant than results.

The author chose to place references at the ends of chapters. The reviewer would have preferred a collected bibliography at the end of the text but this is a personal preference and in any event a very mild carp at such an excellent addition to the field.

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129[K].—ULF GRENANDER & MURRAY ROSENBLATT, *Statistical Analysis of Stationary Time Series*, John Wiley & Sons, New York, 1957, 300 p., 23.5 cm. Price $11.00.

This book is the first up-to-date extended treatment of a class of problems that for many years have interested (and frequently baffled) workers in a wide range of fields. Aside from its broad mathematical interest the book will be of significance for numerical analysts who are called upon to grapple with real data. The authors are to be congratulated, not only for bringing together in readable form the widely scattered existing material, but also for their own substantial contributions to the field.

A sketch of the contents is the following: The first two chapters are introductory in nature, setting forth the necessary probability theory and related concepts, as developed by Kolmogoroff, Cramer, Doob, Karhunen and others; Chapter 3 surveys the analysis of "finite parameter models" that have been widely studied and used over the past quarter-century. Much of the material in the remaining chapters is novel, and due to the authors themselves. Chapter 4 considers methods of estimation of the spectral density; Chapter 5 touches upon applications to analysis of noise and turbulence; Chapter 6 concerns asymptotic distribution theory and confidence bands for the estimates, and contains an interesting account of the analysis performed on several artificially generated time-series; Chapter 7 concerns regression analysis; Chapter 8 deals briefly and sometimes heuristically with certain miscellaneous problems, including the estimation of zeros and extrema of random processes. The book closes with sets of exercises for the several chapters, an ap-

pendix on some complex function theory needed especially in the second chapter, a five-page bibliography and a brief index.

This book is well set-up, and only a few minor misprints were noted.

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130[K, G].—J. ROY & R. G. LAHA, "Classification and analysis of linked block designs", *Sankhya*, v. 17, 1956, p. 115–132.

Youden's linked block (LB) designs are obtained by reversing the roles of blocks and varieties in Balanced Incomplete Block (BIB) designs. We assume v varieties in b blocks, each of k plots ($k < v$), such that each variety occurs at most once in any block and altogether in r blocks. Let μ_{ij} be the number of varieties which occur both in the ith blocks ($i \neq j = 1, 2, \cdots, b$). An LB design has all $\mu_{ij} = \mu$. Most LB designs are Partially Balanced Incomplete Block (PBIB) designs. Plans for certain PBIB designs are given by Bose, Clatworthy and Shrikhande [1].

The general intra-block analysis of variance for LB designs is given in Table 1, followed by a formula for the efficiency factor. Tables 2–5 present a worked-out example of an LB design which is not given in the Bose bulletin.

The authors have derived necessary and sufficient conditions that a Two-Associate PBIB design may be of the LB type. They present in Table 6 a list of LB designs with r, $k \leq 10$; v ranges from 6 to 82 and b from 3 to 64. Values of various parameters, including μ and the efficiency factor to 2D, are given for each design, plus a reference to the Bose bulletin if the design is contained therein or could be inferred from it. The actual experimental plans are given in Tables 7–13 for all designs which could not be secured from the Bose bulletin.

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1. R. C. BOSE, W. H. CLATWORTHY & S. S. SHRIKHANDE, "Tables of partially balanced designs with two associated classes," Univ. of North Carolina, *Tech. Bull.* no. 107, Reprint Series no. 50, 1954.

131[K].—C. S. RAMAKRISHNAN, "On the dual of a PBIB design and a new class of designs with two replications", *Sankhya*, v. 17, 1956, p. 133–142.

The dual of any design is found by interchanging blocks and varieties in the original design. The general analysis of the dual of a two Associated PBIB (Partially Balanced Incomplete Block) design is given in Table 1. A worked-out example is given in Tables 2–6 for a two-replicate dual with 24 varieties (v), 8 blocks (b) and 6 plots per block (k).

Plans and parameters of the four designs of the new class for which $k \leq 10$ are given in Tables 7–9. These designs are PBIB designs with 5 associate classes but are simply analyzed by use of the dual method; $b = 6(2) 12$, $k = b - 2$ and $v = bk/2$.

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132[K].—J. W. Tukey, "Sums of random partitions of ranks", *Ann. Math. Stat.*, v. 28, 1957, p. 987–992.

Suppose the integers $1, \cdots, N$ are randomly distributed among k distinguishable classes with equal probability and without replacement. Let $s_1, \cdots, s_k$ be the sum of the integers in each class and $S = \max(s_1, \cdots, s_k)$. A generating function for the distribution of S in the range of $\frac{1}{k}N(N-1) \leq S \leq \frac{1}{2}N(N-1)$ is obtained. For $k \leq 6$ this is shown to provide the usual percentage points for N up to and beyond 10. Exact tables of 5 % and 1 % points are provided for $k = 2, 3, 6$, and $N = 1(1)10$.

For $k = 2$ the distribution is that of Wilcoxon's paired sample test [2]. For the indicated range of S we have

$$P(S \geq S^* \mid k, N) = k \sum_{i=0}^{N} \binom{N}{i} \frac{(k-1)^i}{k^N} P_W(S \geq S^* \mid i, N - i),$$

where $P_W(S \geq S^* \mid i, N - i)$ is the two-sample Wilcoxon [1] probability, i.e., the probability that the sum of ranks in the first sample will satisfy $S \geq S^*$ where the first and second sample sizes are i and $N - i$.

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1. Evelyn Fix & J. L. Hodges, Jr., "Significance probabilities of the Wilcoxon test", *Ann. of Math. Stat.*, v. 26, 1955, p. 301–311.
2. Frank Wilcoxon, "Probability tables for individual comparisons by ranking methods," *Biometrics*, v. 3, 1947, p. 119–122.

133[K, A].—J. M. Sakoda & B. H. Cohen, "Exact probabilities for contingency tables using binomial coefficients", *Psychometrika*, v. 22, 1957, p. 83–86.

This paper contains a table of all bionomial coefficients $\binom{n}{r}$ to 4D for $n = 1(1)$ 60.

C. C. C.

134[K].—D. B. Owen, *The Bivariate Normal Probability Distribution*, Sandia Corporation Research Report, No. SC-3831 (TR), 1957, 136 p., 22 x 28 cm. Available from the Office of Technical Services, Dept. of Commerce, Washington 25, D. C., Price 65 cents.

See the preceding review for definitions of $B(h, k, \rho)$, $T(h, a)$, $G(h)$ and for some relations among them.

The main table is a 6D table of $T(h, a)$ for $h = 0(.01)3.50(.05)4.75$, $a = 0(.025)$- 1.000, ∞. (For $h > 4.75$, $T(h, a) = 0$ to 6D for all a.) First differences are given for each a and for each h.

Appendix B is a 6D table of $T(0, a) = 1/2\pi$ arc tan a for $a = 0(.005)1.000$. Appendix C is a 7D table of $(1 - \rho^2)^{-\frac{1}{2}}$ for $\rho = 0(.005)1.000$. First differences as given in both tables.

The author suggests that for small h interpolation be performed in terms of $T(0, a) - T(h, a)$. This is equivalent to using the function tabled by C. Nicholson [1]. For large h and linear interpolation the function tabled by Owen appears to be simpler to use.

This report goes into more detail on the derivation of relations among $B(h, k, \rho)$, $T(h, a)$ and $G(h)$ than does the paper reviewed above. It also uses a different approach to the derivation. The change of variable displayed near the top of page 10 works only if values other than principal values are given to arc sin and arc tan for some combinations of values of ρ, k, and h. The resulting form (3.2) is correct.

Owen presents a discussion of interpolation, a table of relations among integrals involving $G(x)$, $G'(x)$, $T(x, y)$, a discussion of the calculation of multivariate normal probabilities and a discussion of other tables available for use in finding bivariate normal probabilities over polygons and ellipses. The reviewer noted two errors in the list of references. The pages for reference 2 are 475–479. The volume and year for reference 3 are III, 1949. In the main table of the paper a misprint appears in a heading on p. 67; 0.01 should read 1.01.

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1. C. NICHOLSON, "The probability integral for two variables," *Biometrika*, v. 33, 1943, p. 59–72.

135[K].—D. B. OWEN, "Tables for computing bivariate normal probabilities," *Ann. Math. Stat.*, v. 27, 1956, p. 1075–1090.

Several tables closely related to the bivariate normal distribution function,

$$B(h, k, \rho) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{k} \int_{-\infty}^{h} \exp\left[-\frac{1}{2} \frac{(x^2 - 2xy + y^2)}{1 - \rho^2} \right] dx\, dy,$$

have been published and others have been calculated. Because of the triple argument extensive tables are required if interpolation is to be relatively easy. Both the paper under review and the one reviewed immediately below contain bibliographies of papers in which authors have suggested and in some cases have tabled functions of two arguments which can be used to obtain values of $B(h, k, \rho)$. Owen suggests using

$$\begin{aligned}
T(h, a) &= \frac{1}{2\pi} \int_0^a \frac{\exp\left[-\frac{1}{2} h^2(1 + x^2) \right]}{1 + x^2}\, dx \\
&= \frac{1}{2\pi} \int_0^{\arctan a} \left(\exp -\frac{1}{2} h^2 \sec^2\theta \right) d\theta . \\
&= \frac{\arctan a}{2\pi} - \frac{1}{2\pi} \int_0^h \int_0^{ax} \exp\left[-\frac{1}{2}(x^2 + y^2) \right] dy\, dx
\end{aligned}$$

In terms of this function and the well tabled

$$G(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} \exp\left(-\frac{1}{2} x^2 \right) dx,$$

$$\begin{aligned}
B(h, k, \rho) &= T\left(h, \frac{k}{h} \right) + T\left(k, \frac{h}{k} \right) - T\left(h, \frac{k - \rho h}{h\sqrt{1 - \rho^2}} \right) \\
&\quad - T\left(k, \frac{h - \rho k}{k\sqrt{1 - \rho^2}} \right) + G(h)G(k) \\
&= \frac{1}{2} G(h) + \frac{1}{2} G(k) - T\left(h, \frac{k - \rho h}{h\sqrt{1 - \rho^2}} \right) - T\left(\frac{h - k}{k\sqrt{1 - \rho^2}} \right) - \frac{\delta}{2}
\end{aligned}$$

Where $\delta = 0$ if $hk > 0$ or if $hk = 0$ and $h + k \geq 0$ and $\delta = 1$ otherwise. Since $T(h, a) = T(-h, a) = -T(h, -a)$ and $T(h, a) = \frac{1}{2}G(h) + \frac{1}{2}G(ah) - G(h)G(ah) - T(ah, 1/a)$ if $a > 0$, $T(h, a)$ is tabled only for $h \geq 0$ and only for $0 \leq a \leq 1$ and $a = \infty$.

Table A gives $T(h, a)$ to 6D for $h = 0(.01)2.00(.02)3.00$, $a = 0(.25)1.00$, Table B gives $T(h, a)$ to 6D for $h - 0(.25)3.00$, $a = 0(.01)1.00$, ∞, Table C gives $T(h, a)$ to 6D for $h = 3.00(.05)3.50(.10)4.70$, $a = 0.10$, $.20(.05).50(.10).80$, 1.00, ∞. For $h \geq 4.76$, $T(h, a) = 0$ to 6D for all a.

Tables A and B will be seen to provide values of $T(h, a)$ on the sides of rectangles in the (a, h)-plane. Owen suggests an interpolation scheme which is linear in the eight values of T, the four values at (h_0, a_0), (h_0, a_1), (h_1, a_0), (h_1, a_1), the corners of the rectangles enclosing the subject (h, a) and the four values at (h_0, a), (h_1, a), (h, a_0), (h, a_1). Linear interpolation may be necessary to find the latter four values. Table D gives empirically determined maximum errors occurring in various ranges when the suggested scheme is used. The highest absolute value listed is $.000,007,1$ which occurs in the range $.50 < h$ $.75$, $.75 < a < 1.00$.

Many of the entries in the tables are also given in the table by the same author which is reviewed immediately below. A comparison of the common entries revealed four last place errors in the present table, all for $a = .75$. $T(.02, .75) = .102393$, $T(.21, .75) = .099819$, $T(.42, .75) = .092421$, $T(.53, .75) = .086974$.

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136[K].—HARRY WEINGARTEN, "Tables for type A critical regions," *Ann. Math. Stat.*, v. 28, 1957, p. 1052–1053.

Let X, Y be independent unit normal random variables, and let A be such that the probability is α that (X, Y) falls above $y = A - Bx^2$. If c is the upper $\alpha/2$ point of the normal distribution, it can be shown that $\sqrt{A/B} \to c$ when $B \to \infty$. This suggests writing $A = c^2B + \rho$. The author tables ρ to 5D for $\alpha = .01$ and $.05$; $B = 0(.1)5(1)10(10)100$. The entries at $B = 0$ are low by one in the last place.

By estimating the integral over the region between the parabola and its tangents at $y = 0$, the reviewer derived the approximation $\rho^* = (1 + c^2)/4c^2B$ for ρ, which should be good for large values of B. A comparison of ρ^* with the tabled values of ρ for $B < 5$ supports the approximation. Surprisingly, the agreement between ρ and ρ^* deteriorates for larger B values. For example, when $\alpha = .05$ and $B = 100$, $\rho = .00401$ and $\rho^* = .00315$. Most of the discrepancy can be explained by the assumption that not enough figures were retained in c^2 when calculating ρ from A.

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137[K].—D. S. PALMER, "Properties of random functions," *Cambr. Phil. Soc.*, *Proc.*, v. 52, 1956, p. 672–686.

This paper considers the density functions of lengths of intervals between successive zeros (Table 5) and maxima (Table 6) and the lengths of intercepts by a

horizontal line of two random functions of known autocorrelations and cross correlation. The tabulated distributions assume Gaussian autocorrelation $\exp(-\tfrac{1}{2}t^2)$ and give both density and cumulative entries for $t = 0(.1)8$, to 3D for the density and to 4D for the cumulative probabilities.

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138[K].—B. I. Hartley & E. S. Pearson, "The distribution of range in normal samples with $n = 200$," *Biometrika*, v. 44, 1957, p. 257–260.

A 7D table is presented of

$$P(W) = Pr(w \leq W) = \left\{ \int_{-\frac{1}{2}w}^{\frac{1}{2}w} z(x)\,dx \right\}^n + 2n \int_{\frac{1}{2}w}^{\infty} z(u) \left\{ \int_{u-w}^{u} z(x)\,dx \right\}^{n-1} du$$

where $z(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}$, $n = 200$, and $W = 3.25(.25)9.75$. This is the interesting range of W since $P(9.75) - P(3.25)$ equals one to 7D.

The computation was motivated by a desire to obtain a foothold in applying the interrelated method of Tukey [1] and was carried out on desk calculators using [2].

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1. J. W. Tukey, "Interpolations and approximations related to the normal range", *Biometrika*, v. 42, 1955, p. 480–485.
2. NBS Applied Mathematics Series, No. 23, *Tables of Normal Probability Functions*. U. S. Gov. Printing Office, Washington, D. C., 1953.

139[K].—S. K. Mitra, "Tables for tolerance limits for a normal population based on sample mean and range or mean range", Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 88–94.

If $\bar{x}$ and r are respectively the observed sample mean and range in a random sample of size n from a normal population, then we may designate by $\bar{x} \pm k_1 r$ the confidence interval which will include at least a proportion p of the population with probability or assurance β. In the article under review, Mitra gives a table (Table I) for the factors k, to 3D, for $p = .75, .90, .95, .99$ and $.999$, for $\beta = .75, .90, .95$ and $.99$, and for sample sizes $n = 2(1)20$. The table is therefore a useful analogue for confidence intervals based on sample range instead of sample standard deviation.

For the confidence interval $\bar{x} \pm k_2 \bar{r}$ based on several samples, two tables are given for the factors k_2 to 3D such that the probability is that at least a proportion p of the population will be included in the above confidence interval, where $\bar{x}$ is the grand mean and $\bar{r}$ the mean range in N samples of size $m = 4$ (Table II) or size $m = 5$ (Table III). The proportions p and assurance probabilities β cover the same numerical values given above, and N covers values $4(1)20(5)30(10)50, 75, 100$ and ∞. Although $m = 7$ or $m = 8$ would lead to shorter confidence intervals, as pointed out by Mitra, the selection of group sizes or sub-sample sizes $m = 4$ and $m = 5$ was based on the most commonly used sample sizes in control chart or quality control work.

With the aid of these tables, tolerance limits can therefore be computed quickly,

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and in quality control or acceptance procedures the "spread" of the lot can thereby be estimated accurately.

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140[K].—J. ST.-PIERRE & A. ZINGER, "The null distribution of the difference between the two largest sample values," *Ann. Math. Stat.*, v. 27, 1956, p. 849–851.

Let u = (largest)-(next to largest) of values of a sample of size $n + 1$ from a normal population with zero mean and unit variance. Let $\Phi_n(u)$ denote the cdf of u. Table I furnishes 5D values of $\Phi_n(u)$ for $n = 2(1)7$ and $u = .0(.2)2.6$.

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141[K].—A. E. SARHAN & B. G. GREENBERG, "Estimation of location and scale parameters by order statistics from singly and doubly censored samples, Part I. The normal distribution up to samples of size 10." *Ann. Math. Stat.*, v. 27, 1956, p. 427–451.

This paper is concerned with estimating parameters of a normal distribution from Type II singly and doubly censored samples using order statistics. As distinguished from a Type I censored sample in which points of censoring are fixed, a Type II censored sample is one in which the total number of sample observations as well as the number of censored observations in each censored category is fixed. Five tables to facilitate computation are included.

Table I consists of variances and covariances to 10D of order statistics in samples from a normal population for sample sizes $n = 2(1)20$. Table II consists of coefficients to 8D of the most efficient linear systematic estimates of the mean and standard deviation in censored samples of sizes $n = 2(1)10$ from a normal population. Table III consists of variances and covariances to 8D of estimates obtained using the coefficients of Table II. Table IV consists of percentage efficiencies to 2D of estimates based on censored samples relative to uncensored samples, for $n = 1(1)10$. Table V consists of variances to 8D and relative efficiencies to 2D of alternative estimates proposed by Gupta [1] for the mean and standard deviation. In this table, $n = 10$, $r_1 = 0(1)4$, $r_2 = 0(1)8$, where r_1 and r_2 are the numbers of censored samples in the left and right tails respectively.

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1. A. K. GUPTA, "Estimation of the mean and standard deviation of a normal population from a censored sample", *Biometrika*, v. 39, 1952, p. 260–273.

142[K].—P. M. GRUNDY, M. J. R. HEALY, & D. H. REES, "Economic choice of experimentation," Roy. Stat. Soc., *Jn.*, sec. B, 1956, p. 32–55.

Suppose the new process gives a true increase in output θ for a specified scale of production—so that, if it were adopted for full-scale use, the resulting increase in output would be $k'\theta$ where k' is a constant, supposed known. A first set of n_1 experi-

ments is to be carried out, and then if necessary a second set of n_2 experiments. It is assumed that for fixed n_1 and n_2 these provide independent and normally distributed estimates x_1, x_2 of θ with variances σ^2/n_1 and σ^2/n_2—where σ^2 is known from previous experience, or else can be estimated from the results themselves with sufficient precision, so that errors of estimation of σ^2 can be ignored. The cost of the second set of experiments is taken to be kn_2, where k is a known constant and is measured in the same units as k'. It is convenient to regard n_2 as continuously variable ($\geqq 0$), the value 0 covering the case when a decision is reached at the end of the first set of experiments. The basic problem is then to determine an optimum value of n_2 in terms of the known quantities n_1, x_1, k, k' and $\sigma^2 (n_1, k, k', \sigma > 0)$.

The decision will naturally be based on the sign of the mean of all experiments

$$\frac{(n_1 x_1 + n_2 x_2)}{n_1 + n_2}.$$

For given values of x_1, n_2 and θ, there will be a certain probability P of getting a positive mean and so of accepting the new treatment, so that a gain of $Pk'\theta$ on the average is expected. Against this must be set the cost, kn_2, of the experiments, so that the cost minus expected gain is given by

$$R(\theta, x_1, n_2) = kn - Pk'\theta. \tag{1}$$

This is termed the conditional risk (given θ, x_1, n_2), measured from the status quo. P is in fact the probability (given θ, x_1 and n_2), that x_2 exceeds $-n_1 x_1/n_2$, which gives

$$P = \varphi \mid (n_1 x_1 + n_2\theta)/n_2^{\frac{1}{2}} \mid \tag{2}$$

where φ denotes the normal integral,

$$\varphi(t) \equiv (2_\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} \exp\left(-\frac{1}{2}x^2\right) dx.$$

The risk function $R(\theta)$ is the expected value of $R(\theta, x_1\, n_2)$. It is desired to minimize $R(\theta)$, so far as the conflicting needs of different values of θ permit it, so it is reasonable to attempt to minimize $R(\theta, x_1, n_2)$. This, however, does not provide an immediate solution to the problem because θ is unknown, so that the quantity obtained by integrating R over the fiducial distribution of θ based on the estimate x_1 is minimized. This distribution is normal with mean x_1 and variance σ^2/n_1, so that the resulting quantity, which is called the integral risk is

$$\bar{R}(x_1, n_2) = \int_{-\infty}^{\infty} r(\theta, x_1, n_2)\, \varphi'\left[\frac{\theta - x_1}{n_1^{\frac{1}{2}}}\right] \frac{n_1^{\frac{1}{2}}}{\sigma}\, d\theta. \tag{3}$$

The integral risk of equation (3) is given by

$$\bar{Q}(X, N) = N - \lambda\{X\Phi[X((N+1)/N^{\frac{1}{2}}] + (N/(N+1))^{\frac{1}{2}}\Phi'[X((N+1/N)^{\frac{1}{2}}]\}. \tag{4}$$

where

$$X = \frac{x_1\sqrt{n_1}}{\sigma}, \quad N = n_2/n_1, \quad \text{and} \quad \lambda = \frac{k'\sigma}{kn_1^{3/2}}.$$

For fixed λ, n_2 is to be chosen to minimize $\bar{Q}$.

Table 1 gives values of the integral gain (corresponding to the expression in curly brackets in (4)) measured in terms of $k'\sigma/\sqrt{n_1}$ so that the minimum of (4) can be evaluated using bivariate interpolation. Values of the integral gain are given to 6D for $X = -2.5, -2.(.25)0$ and $N = .5(.5)3., 4., 5., 6., 8., 10, 15, 20, \infty$.

To avoid bivariate interpolation in Table 1, the integral gain can be expressed as

$$\frac{k'\sigma}{n_1^{\frac{1}{4}}} \left(\frac{N}{N+1} \right)^{\frac{1}{4}} \omega(\eta)$$

where $\eta = X((N+1)/N)$ and $\omega(\eta)$ is given in table 2 to 6D for $\eta = .0(.1)4.4$ together with its first two derivatives to 5D and 4D respectively. A monogram for determining the proper value of N has been constructed using the scales of X, λ, and N. Finally, tables are provided for judging the performance of the suggested rule using as measures the probability of reaching a correct decision.

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143[K].—D. J. FINNEY, "The consequences of selection for a variate subject to errors of measurement," Inst. International de Stat., *Revue*, v. 24, 1956, p. 1–10.

Let $y = x + w$, where w is normally distributed and is independent of x. Values of y are observed, and a fraction P of the largest values of y is selected. The problem is to determine the conditional distribution of x given that y is in this fraction. Using a formula of Bartlett [1] for the characteristic function of a conditional statistic and the method of Cornish and Fisher, expressions for the first four moments are given. For the case where x is normally distributed, tables are provided to facilitate computations. Let

$$P = P(T) \equiv \int_T^\infty (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \, dx; \quad Z = Z(T) \equiv dP/dT; \quad \nu = Z/P.$$

Table 1 gives values of T, Z, ν', ν'', to 8D; ν''', ν'''' to 6D for $P = .001, .005, .01, .0125, .02, .025(.025).10(.05).95$. Table 2 gives values of certain functions of T for these values of P.

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1. M. S. BARTLETT, JR., "The characteristic function of a conditional statistic", Lond. Math. Soc., *Jn.*, v. 13, 1938, p. 62–67.

144[K].—TOSIO KITAGAWA & TSUNETAMI SEGUCHI, "The combined use of runs in statistical quality controls," *Bull. Math. Stat.*, v. 7, 1956, p. 25–45.

The authors propose control charts for sample means in which both limit lines and the occurrence of runs are used as criteria for rejection of the hypothesis that the samples of n are drawn from a normal universe with mean m and variance σ_0^2 (the standard given case in quality control terminology). For the use of runs of length 2, real numbers, a_1, a_2 such that $0 \le a_2 \le a_1 < \infty$, are suitably chosen, and the intervals, $(0 < y < a_2)$, $(-a_2 < y \le 0)$, $(y \le a_1)$, $(y \le -a_1)$,

$(a_2 \leqq y < a_1)$ and $(-a_1 < y \leqq -a_2)$ are labeled E_{+0}, E_{-0}, E_{+1}, E_{-1}, E_{+2}, E_{-2} respectively with $E_j = E_{+j} UE_{-j}(j = 0, 1, 2)$. Let y_i be an observed value of the random variable

$$Y_i = \frac{\sum_{j=1}^{n} x_{ij}}{n} - m$$

in which each x_{ij} is $N(m + k\sigma_0, \sigma_0^2)$, $(i = 1, 2, 3, \cdots ; j = 1, \cdots n$. Then the hypothesis that the x's are so distributed with $k = 0$ is rejected on the first occurrence, say for y_n, of either of the events $A_1: y_n \epsilon E_1$ or $A_2: y_{n-1} \epsilon E_2$ and $y_n \epsilon E_1 UE_2$. This is called the "statistical controlling method," $C(1, 2)[a_1, a_2]$. Three other statistical control methods, $C^{(+)}(1, 2)$, $C^{(-)}(1, 2)$, $C^{(\pm)}(1, 2)$ are obtained by replacing the E_j's by E_{+j}'s, by E_{-j}'s, and by the union of these two regions of rejection.

Provision for the use of runs of lengths 2 and 3 is made by the choice of the real numbers a_1, a_2, a_3 with $0 \leqq a_3 \leqq a_2 \leqq a_1 < \infty$ and the designation of 8 intervals, $(0 < y < a_3)$, $(-a_3 < y \leqq 0)$, $(y_1 \geqq a_1)$, $(y \leqq -a_1)$, $(a_2 \leqq y < a_1)$, $(-a_1 < y \leqq -a_2)$, $(a_3 \leqq y < a_2)$, and $(-a_2 < y \leqq -a_3)$ labeled E_{+0}, E_{-0}, E_{+1}, E_{-1}, E_{+2}, E_{-2}, E_{+3} and E_{-3} respectively with again $E_j = E_{+j} UE_{-j}$ $(j = 0, 1, 2, 3)$. Then the control method $C(1, 2, 3)$ is obtained by rejecting the same null hypothesis on the first occurrence of one of the (not mutually exclusive) events: $B_1: y_n \epsilon E_1$; $B_2: y_{n-1} \epsilon E_2$ and $y_n \epsilon E_1 UE_2$; $B_3: y_{n-z} \epsilon E_3$, $y_{n-1} \epsilon E_2 UE_3$ and $y_n \epsilon E_1 UE_2 UE_3$; $B_4: y_{n-2} \epsilon E_2$, $y_{n-1} \epsilon E_3$ and $y_n \epsilon E_1 UE_2 UE_3$. It is noted that for five special cases: (1) $a_3 = a_2$; (2) $a_2 = a_1$; (3) $a_3 = a_2 = a_1$; (4) $a_3 = a_2$ and $a_1 = \infty$; (5) $a_1 = a_2 = \infty$, $C(1, 2, 3)$ reduces to $C(1, 2)$, $C(1, 3)$, $C(1)$, $C(2), C(3)$ respectively in which the last three are ordinary control chart methods. Again $C^{(+)}(1, 2, 3)$, $C^{(-)}(1, 2, 3)$ and $C^{(\pm)}(1, 2, 3)$ are defined as in the case of runs of length 2 only.

Let $T(i)$, $T^{(+)}(i)$, $T^{(\pm)}(i)$, $T(ij)$, $T^{(+)}(ij)$, $T^{(\pm)}(ij)$, etc. be the expected number of samples of n required for rejection under the corresponding control methods for a given value of k. The authors develop means for calculating the appropriate probabilities and expected values, and then for samples of $n = 4$ and 10 and $a_1 = 3.09\sigma_0/\sqrt{n}$, $a_2 = 1.85\sigma_0/\sqrt{n}$, $a_3 = 1.26\sigma_0/\sqrt{n}$ give the following tables for $k = 0(.2)2$; (1) The probabilities to $4D$ for a sample mean to fall in each of the 8 intervals $E_j(j = -3, -2, \cdots, 3)$; (2) The values of $T(1)$, $T(2)$, $T(3)$, $T(1, 2)$ $T(1, 2, 3)$, $T^{(\pm)}(1)$, $T^{(\pm)}(2)$, $T^{(\pm)}(3)$, $T^{(\pm)}(1, 2)$, $T^{(\pm)}(1, 2, 3)$, $T^{(+)}(1)$, $T^{(+)}(2)$, $T^{(+)}(3), T^{(+)}(1, 2)$ and $T^{(+)}(1, 2, 3)$ all to $2D$.

A second set of alternate hypotheses H_l^*, is considered: $k = 0$ but $\sigma_0^2 = l^2\sigma_0^2$;— the first set being designated by H_k. Then for each of the five control methods: (I) $C(1)$ [3], the usual 3-σ control method; (II) $C(1, 2)$ [3.205, 2.067]; (III) $C^{(\pm)}(1, 2)$ [3.205, 1.927]; (IV) $C(2)$ [1.932]; (V) $C^{(\pm)}(2)$ [1.781], the reciprocals of the corresponding T functions are given to $5D$ for $k = 0(.2)2$ for H_k and for $l = 1(.25)3$ for H_l^*. Since for all of these control methods T_0^{-1} $(k = 0) = 0.00270$ approximately these last two tables enable one to compare the power functions of the five control methods.

C.C.C.

145[K].—Tosio Kitagawa & Tsunetami Seguchi, "The combined use of runs in statistical quality controls. II," *Bull. Math. Stat.*, v. 7, 1956, p. 53–72.

This paper is a continuation of a paper reviewed above and is largely concerned with numerical results for the control methods $C(1)$ $[a_1]$, $C(1, 2)$ $[a_1, a_2]$, $C^{(\pm)}(1, 2)$ $[a_1, a_2]$, $C(2)$ $[a_2]$ and $C^{(\pm)}(2)$ $[a_2]$ which throw further light on their power functions. As in the first paper two sets of alternate hypotheses are considered, H_k and H_l^*; the null hypothesis being specified by $k = 0$ or $l = 1$. For $C(1)$, $C(2)$ and $C^{(\pm)}(1, 2)$, Table I(1) gives the values of a_1 or a_2 to 3D to make α, the risk of error of the first kind, be .01 and .001. For $C(1, 2)$ and $C^{(\pm)}(1, 2)$, let $\rho = T_0(1)/T_0(2)$ or $T_0^{(\pm)}(1)/T_0^{(\pm)}(2)$ where $T_k(1)$, etc., is the expected number of samples of n required to reject the null hypothesis for $C(1)$, etc. Then Table I(2) gives values of a_1 and a_2 to 3D to make $\alpha = .01$ or .001 for $\rho = .5, 1, 2$ for $C(1, 2)$ and $C^{(\pm)}(1, 2)$. With these a_i's $(i = 1, 2)$ Tables II gives to 5D the fundamental probabilities of samples falling into the intervals defined by the a_i's for $C(1)$, $C(2)$, $C^{(\pm)}(2)$, $C(1, 2)$ and $C^{(\pm)}(1, 2)$, $\alpha = .01, .001$, $k = 0(.2)3$ or $l = 1(.25)3$ and $\rho = .5, 1, 2$. Tables III give to 5D the reciprocals of the number of samples required to reject the null hypothesis for the same control methods and values of the parameters as in the Tables II. Finally Tables IV give to 3D the ratio of the reciprocals given in Tables III for each of the other control methods to the corresponding figure for $C(1)$.

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146[K].—F. C. Leone, R. B. Nottingham, & Jack Zucker, "Significance tests and the dollar sign," *Industrial Quality Control*, v. 13, no. 12, 1957, p. 5–21.

Twelve nomograms are given for controlled processes obeying a normal frequency law. Let μ be the mean of the new process, μ_0 the mean of the present process, σ^2 the variance of the new process, σ_0^2 the variance of the present process. Nomograms 1–3 give 90 per cent, 95 per cent, and 98 per cent confidence limits on $(\mu - \mu_0)/\mu_0$. Nomograms 4–6 give 90 per cent, 95 per cent, 98 per cent confidence limits on $1 - (\sigma^2/\sigma_0^2)$.

Let σ_1^2 be the variance of the process with the larger sample variance, σ_2^2 the variance of the process with the smaller sample variance. Nomograms 7–8 give the lower and upper 90 per cent confidence limits on σ_1^2/σ_2^2, nomograms 9–10 the 95 per cent limits, nomograms 11–12 the 98 per cent limits.

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147[K].—R. M. Kozelka, "Approximate upper percentage points for extreme values in multinomial sampling," *Ann. Math. Stat.*, v. 27, 1956, p. 507–512.

This paper does not deal with extreme values but with largest frequencies. The social scientists are interested in a method of testing the largest observed frequency in a k fold multinomial distribution for the null hypothesis of equal probability for each category. Let $F_i(i = 1, 2, \cdots, k)$ be the observed proportions of a sample of n objects into k multinomial categories with equal probabilities. Then it is assumed that the joint distribution of the observed proportions is asymptotically multi-

variate normal. The author calculates the normal approximation $P(\max t_i \geqq t^*)$ for the standardized variate $t_i = Kn(kF_i - 1)/(k - 1)$ and for large t. Table 1 gives the critical values of t for .95 and .99, significance levels for $k = 1(1)\ 25$ to 3D. Table 2 compares the computed and approximated upper percentage points for $k = 3, 4, 5$ and $n = 3\ (1)\ 12$. The approximation is satisfactory even for small n. However, it decreases in accuracy for increasing k. Only for the case $k = 3$, the approximate density provides a suitable approach to the moment generating function.

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148[K].—G. S. JAMES, "On the accuracy of weighted means and ratios", *Biometrika*, v. 43, 1956, p. 304–321.

This paper contains tables of the upper 5 %, $2\frac{1}{2}$ %, 1 % and $\frac{1}{2}$ % critical points of the distribution of the statistic, $u = (\hat{x} - \mu) . \sqrt{(w_1 + w_2)}$, where $\hat{x} = (w_1 x_1 + w_2 x_2)/(w_1 + w_2)$ is the weighted mean of two independent normally distributed variables x_1 and x_2 having the same expected value μ and variances $\lambda_1 \sigma_1^2$ and $\lambda_2 \sigma_2^2$ respectively. The value of μ, σ_1^2 and σ_2^2 are assumed to be unknown; the values of λ_1 and λ_2, known. There are available independent estimates, s_1^2 and s_2^2, of σ_1^2 and σ_2^2, based on ν_1 and ν_2 degrees of freedom respectively. The weights used are $w_1 = 1/\lambda_1 s_1^2$ and $w_2 = 1/\lambda_2 s_2^2$. Thus, for example, if x_i and s_i denote the sample means and variance estimates from two samples of size n_i $(i = 1, 2)$ taken from two normal populations, with the same mean μ, then, taking $\nu_i = n_i - 1$ and $\lambda_i = 1/n_i$, the tables may be used to place confidence limits on μ.

The following values are tabulated: (here $r = w_1/[w_1 + w_2]$), all to 2D: upper 5 % and $2\frac{1}{2}$ % critical values of u for $r = 0(.1)1$, ν_1, $\nu_2 = 6, 8, 10, 15, 20, \infty$ and upper 1 % and $\frac{1}{2}$ % values for the same values of r but for ν_1, $\nu_2 = 10, 12, 15, 20, 30, \infty$.

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149[K].—IRWIN GUTTMAN, "On the power of optimum tolerance regions when sampling from normal distributions," *Ann. Math. Stat.*, v. 28, 1957, p. 773–778.

The author determines the power function of optimum β-expectation tolerance regions [1], when sampling from normal distributions. In all cases the author uses a measure of desirability of .99. When sampling from a normal population $N(\mu, \sigma^2)$, in Case I: μ, σ^2 unknown, the author lists the power to 4D for $a = .870167, .760906, .638572$, and $.446594$, $\beta = .975, .95, .90, .75$ and sample sizes $n = 2(1)5, 7, 11(10)$ 41, 61 and 121. The alternative to σ^2 is $\alpha^2 \sigma^2$, $0 < \alpha < 1$. In Case II, the mean is unknown, the variance is known; and in Case III, the mean is known, the variance unknown. In Case II, the power is tabulated for the same values of α, β, and n as in Case I. In Case III, the values of n are $1(5), 7, 11, 21, 40, 60$ and 120. Consider sampling from a multivariate population

$$c \exp\left[-\tfrac{1}{2}(\omega - \mu)\Lambda(\omega - \mu)'\right], \quad \mu = (\mu_1, \cdots, \mu_k), \quad \Lambda^{-1},$$

the variance covariance matrix of $\omega = (X_1, \cdots, X_k)$, and the alternative to Λ^{-1}

is $\alpha^2 \Lambda^{-1}$, $0 < \alpha < 1$, the author finds the power function. The power is listed for $\alpha = .88927, .79697, .69432, .53403$; $\beta = .925, .95, .90, .75$, and $n = 3, 4, 5, 7, 11,$ 21, 30, 31, 32 in the case of a bivariate normal population with $\rho = 0$. The power is given to 4D in all cases.

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1. D. A. S. FRASER & IRWIN GUTTMAN, "Tolerance regions," *Ann. Math. Stat.*, v. 27, 1956, p. 162–179.

150[K].—S. S. GUPTA & MILTON SOBEL, "On a statistic which arises in selection and ranking problems", *Ann. Math. Stat.*, v. 28, 1957, p. 957–167.

Let $y = (x_{[p]} - x)/s$ where $x_{[p]}$, x, and s^2 are mutually independent random variables with the following properties: (a) $x_{[p]}$ is the largest in a sample of p from a normal distribution with mean μ and variance σ^2, (b) x is an observation from the same normal distribution, and (c) $\nu s^2/\sigma^2$ has a χ^2 distribution with ν degrees of freedom. A 2D table is given of $y_p{}^*$ where $P(y \le y_p{}^*) = P^*$ for $P^* = .75, .90, .95,$ .975, and .99 for $k = p + 1 = 2, 5, 10(1)\ 16, 18, 20(5)\ 40, 50$, and $\nu = 15(1)\ 20,$ 24, 30, 36, 40, 48, 60, 80, 100, 120, 360, and ∞. References to related tables are given in section 2 and computation methods are discussed in sections 4, 5, and 6.

<div align="right">I. R. SAVAGE</div>

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151[K].—SEYMOUR GEISSER, "The modified mean square successive difference and related statistics," *Ann. Math. Stat.*, v. 27, 1956, p. 819–824.

Let $x_1, \cdots, x_{2m}$ be independent and identically distributed random variables, each having a $N(0, \sigma^2)$ distribution, and consider the statistics $\delta_0{}^2 = \sum_{i=1, i \neq m}^{2m-1}$, $(x_{i+1} - x_i)^2/4(m - 1)$ and $\xi = \sum_1^{2m} x_i/\delta_0{}^2$. The table gives the values of z for which

$$P(\delta_0{}^2/\sigma^2 \le z) = .025, \quad P(\delta_0{}^2/\sigma^2 \ge z) = .025,$$

$P(|\xi| > z) = .05$ to 3D for $n = 2m = 4(2)50$. An approximation is used when $n > 20$. The asymptotic distribution of each statistic is discussed. The paper is closely related to that of Kamat [1].

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1. A. R. KAMAT, "Modified mean square successive difference with an exact distribution," *Sankhya*, v. 15, 1955, p. 295–302.

152[K].—A. A. ANIS, "On the moments of the maximum of partial sums of a finite number of independent normal variables," *Biometrika*, v. 43, 1956, p. 79–84.

This paper includes a table to 4D of the first four moments of the maximum of $X_1, X_1 + X_2, \cdots, X_1 + X_2 + \cdots + X_n$, where X_i are dependent standard normal variates, for $n = 2(1)\ 15$.

<div align="right">F. J. MASSEY, JR.</div>

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153[K].—P. G. Moore, "The transformation of a truncated Poisson distribution," *Skandinavisk Aktuarietidskrift*, v. 39, 1956, p. 18–25.

To estimate the parameter, λ, of a truncated Poisson distribution, the author proposes the use of Anscombe's transformation, $y \sqrt{x + \frac{3}{8}}$, in which x is the Poisson variable. Then the y's are approximately normally distributed and if $\bar{y}$'s are values of y's referred to the point of truncation as origin, then following HALD [1] the maximum likelihood estimate of the expected value, ξ, of y, is the solution of the equation: $\theta = \frac{1}{2}[\psi'(2\xi) + 2\xi]$ in which $\psi'(z)$ is the logarithmic derivative of cumulative normal frequency function in standard units and θ is the mean of the observed $\bar{y}$'s. Table 1 gives values of ξ to 3D for $\theta = 0.1(.01)1.14 (.02)1.54$. Table 2 gives n var (ξ) for samples of n, to 4D for $\xi = -.5(.05)1.5$. Table 3 gives values of ξ' to 4D for values of $\lambda = .6(.1)6(.25)9$ where $\xi' = \xi +$ the point of truncation. If the observed distribution is censored instead of being truncated, another transcendental equation, also given by HALD [1] connects φ, the average of the known $\bar{y}$'s, ξ and p, the portion of the distribution that is observed. Fig. 1 provides a graphical solution for p and ξ by reading values of ξ in the range $(-1, 1.5)$ and φ in the range $(-1, 6)$ for $p = .2(.2)1$. Table 4 gives in this case values of n var ξ to 4D for $\xi = -.5(.1)1.4$. The reader should note that the expression for the sample average of the $\bar{y}$'s is incorrectly written in the text.

C.C.C.

1. A. HALD, "Maximum likelihood estimation of the parameters of a normal distribution which is truncated at a known point," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 119–129.

154[K].—Jogabrata Roy & S. K. Mitra, "Unbiassed minimum variance estimation in a class of discrete distributions," *Sankhyā*, v. 18, 1957, p. 371–378.

Consider the Poisson distribution truncated at zero: $P(X = x) = e^{-\lambda x}\lambda/x!$ $(1 - e^{-\lambda})$, $x = 1, 2, \cdots$. If T is the sum of n independent observations on this distribution, then the minimum variance unbiased estimate for λ is $T u(T - 1, n)/u(T, n)$ where the u are Stirling's numbers of the second kind. The authors give a 3D table of the estimate for $n = 2(1)10$. In casual checking I noted these errors:

n	t	correct value
8	12	.896
8	71	8.873
9	11	.429
9	32	3.455
9	20	1.906
9	44	4.857
10	32	3.062

There are several recent unpublished tables of u. Miksa has placed in the UMT File an exact table up to 50 [MTAC, v. 9, 1955, RMT **85**, p. 198], on the basis of which R. F. Tate, University of Washington, has prepared a 5D table of nu $(t - 1, n)/u(t, n)$; a version of this table is to appear in the September 1958 issue of the *Annals of Mathematical Statistics*. Richard T. Burch, Department of Defense, programmed the computation of a 4S table of u for $t \leq 100$.

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155[K].—G. J. RESNIKOFF & G. J. LIEBERMAN, *Tables of the non-central t-Distribution*, Stanford University Press, Stanford, Calif., 1957, 389 p., 24 x 17 cm. Price $12.50.

The probability density for the non-central t statistic is given by

$$h(f, \delta, t) = \frac{f!}{2^{\frac{f-1}{2}} \Gamma\left(\frac{f}{2}\right) \sqrt{\pi f}} e^{-\frac{1}{2}\delta^2/(f+t^2)} \frac{f^{(f+1)/2}}{f+t^2} Hh_f\left(\frac{-\delta t}{\sqrt{f+t^2}}\right)$$

where

$$Hh_f(y) = \int_0^\infty \frac{v^f}{f!} e^{-\frac{1}{2}(v+y)^2} dv$$

and

$$t = \frac{z + \delta}{\sqrt{w}}$$

The quantity z is a normally distributed random variable with zero mean and unit standard deviation, w is a random variable distributed independently of z as χ^2/f, and f is the number of degrees of freedom.

The general usefulness of "Students" t statistic and related statistical analyses are well known and well recognized. The present tables of the non-central t or the power function of Students' t-test have rather general and wide application. Indeed, in addition to finding the chance of rejecting the null-hypothesis of Students' t-test when some alternative is true, the present tables find application to establishing confidence intervals for proportions from a normal population. They are needed for sampling inspection by variables for fraction defective with one or two standards given; they are used in the Wald, Arnold, Goldberg and Rushton (WAGR) sequential test for the proportional of a normal population exceeding a given value, and also even in a test of the hypothesis that the coefficient of variation of a normal population is a stated value along with the power of the test and other important uses. The publication of these tables therefore require no justification, but rather are a most welcome addition to existing statistical aids to computation and analysis.

The tables include a table of values to 4D of the probability density function $h(f, \delta, t)$ of the non-central t as a function of $t/\sqrt{f}$, a table to 4D of the probability integral $P(f, \delta, x) = \text{Pr. } \{t/\sqrt{f} \le x \mid f, \delta\}$, and a table to 3D of percentage points of the non-central t-statistic, i.e. values of a such that $P_r(t/\sqrt{f} > x) = \epsilon$. The argument $t/\sqrt{f}$ is used instead of t since the range of $t/\sqrt{f}$ is roughly the same whatever the values of f and δ, thus resulting in more compact tables. The degrees of freedom, f, cover the range $2(1)24(5)49$ and the non-centrality parameter, δ, is accounted for by the relation $\delta = \sqrt{f+1} \, K_p$, where K_p is the standardized normal random variable exceed with probability p, with p having the values .25, .15, .1, .065, .04, .025, .01, .004, .0025 and .001 in the tables.

These tables of the non-central t-distribution are comprehensive, well-organized and apparently fill the needs for most practical applications in statistical analysis. The authors, Resnikoff and Lieberman, are to be praised for the excellent and very

complete Introduction, for it is most instructive both in theory and in the applications to important practical examples.

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156[K].—A. KUDO, "Tables for studentization," *Sankhyā*, v. 15, 1957, p. 163–166.

Let $(x_1, \cdots, x_k)$ be independent and identically distributed $N(O, \sigma^2)$ random variables, s^2 and y be independent statistics, where ns^2/σ^2 has a χ_n^2 distribution, and suppose the distribution function $F(x)$ of y/σ is known. The problem is to determine the distribution function $F_n(x)$ of y/s. An approximation to $F_n(x)$ in terms of the derivatives of $F(x)$ is given by

$$F_n(x) = \sum_{m=1}^{N} b_m(n) x^m F^{(m)}(x) + R_N ,$$

where

$$b_0(n) = 1,$$

$$b_m(n) = \sum_{r=0}^{m} \frac{(-1)^{m-r}}{r!(m-r)!} \left(\frac{2}{n}\right)^{r/2} \frac{\Gamma\left(\dfrac{n+r}{2}\right)}{\Gamma\left(\dfrac{n}{2}\right)} ,$$

and R_N is the remainder. The table gives the values of $b_m(n)$ to 7D for $n = 1(1)$ 20 (5) 100, $m = 2(1)$ 5; $n = 1(1)$ 20 (5) 35, $m = 6(1)$ 9; $n = 1, 2, m = 10(1)$ 13. The computation was based on writing $b_m(n)$ as a function of n and $C(n) = (2/n)^{\frac{1}{2}}\Gamma([n+1]/2)/\Gamma(n/2)$, and evaluating $C(n)$ from the National Bureau of Standards Tables [1] and a certain expansion.

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1. NBS Applied Mathematics Series, no. 16, *Tables of n! and* $\Gamma(n + \frac{1}{2})$ *for the First Thousand Values of r*, U. S. Gov. Printing Office, Washington, D. C., 1951.

157[K].—W. L. JENKINS, "An improved method for tetrachoric *r*," *Psychometrika*, v. 20, 1955, p. 253–258.

A procedure and accompanying tables are given for correcting tetrachoric correlation, *r*, as follows:

Letter the tetrachoric frame $\begin{pmatrix} c & d \\ a & b \end{pmatrix}$ so that $a < d$ and $ad > bc$ and compute the ratio ad/bc. Then compute left and lower marginal splits $(a + b)/$total and $(a + c)/$ total to be used for base correction given in table 2 to 3D for the smaller split = .1 (.01).2 (.02).46 and the larger split = .1 (.02).8. (Table 1 shows uncorrected *r* to 3D related to cross product ratios with *r* = .05, (.05),.95.) Table 3 shows a multiplier to 2 or 3D to apply to the entries of table 2 for uncorrected *r* = .1 (.02).96. There alternative selections are made according to the values of the larger marginal split. If the larger split is $\leq$ .4 the choice is made by difference of splits = 0(.05).3 but if the larger split is $>$ .4. the choice is by the smaller split = .1(.1).5. Thus the uncorrected *r* of table 1 is reduced by a value determined by the product of base

correction of table 2 and the multiplier of table 3. The three tables are derived from approximation and adjustments of Pearson's Tables of Normal Correlation Surfaces.

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158[K].—I. R. SAVAGE, "Contributions to the theory of rank order statistics—the two sample case," *Ann. Math. Stat.*, v. 27, 1956, p. 590–615.

Consider two independent samples from two continuous populations and let $F(x)$ denote the *cdf* of the first population and $G(x)$ the *cdf* of the second population. The sample from the first population is of size m and that from the second population is of size n. Let these sample values be pooled and ordered from the smallest to largest. In the resulting ordering replace all values from the first sample by 0 and all values from the second sample by 1. This produces a sequence of 0's and 1's which is denoted by $z_1, \cdots, z_{m+n}$. Each of the possible orderings of the m zero's and the n ones has a probability which depends on $F(x)$ and $G(x)$. Table I contains 4D values of these probabilities for $1 \leq m \leq n \leq 5$ and the case where $F(x) = [H(x)]^{\Delta_1}$ and $G(x) = [H(x)]^{\Delta_2}$; here $\Delta_2 > \Delta_1 > 0$ and $H(x)$ is a continuous *cdf*. For this case, the probabilities can be uniquely expressed as a function of $\delta = \Delta_2/\Delta_1$. The values of δ considered are chosen so that it is related to a special test with power $1 - \beta$ at the α level of significance. The rank orders are arranged according to increasing values of $T = \sum_{i=1}^{m+n} \sum_{j=1}^{i} z_j/i$. Table II contains power function values of the test based T (most powerful for δ near 1), the most powerful rank order test, and the C_1 test (asymptotically most powerful) for $m, n = 1(1)5$ and values of δ corresponding to $(\alpha, \beta) = (.10, .50), (.10, 125), (.05, .25), (.05, .05)$, all to 4D.

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159[K].—W. H. TRICKETT, B. L. WELCH, & G. S. JAMES, "Further critical values for the two-means problem," *Biometrika*, v. 43, 1956, p. 203–207.

Tables are given of $\Pr(\nu \geq V(c; \nu_1, \nu_2, \alpha)) = \alpha$ for $\alpha = .005, .025$ where $\nu = (y - \eta)/(\lambda_1 s_1^2 + \lambda_2 s_2^2)$. Here y is a normally distributed estimate of a parameter η with sampling variance $\lambda_1 \sigma_1^2 + \lambda_2 \sigma_2^2$ with λ_1 and λ_2 known positive constants and s_1^2 and s_2^2 are independent estimates of σ_1^2 and σ_2^2 with chi-square distributions with degrees of freedom ν_1 and ν_2 respectively and independent of y. The tables cover $\nu_1, \nu_2 = 8, 10, 12, 15, 20, \infty$ for $c = \lambda_1 s_1^2/(\lambda_1 s_1^2 + \lambda_2 s_2^2) = 0(.1)1$ and are to 2D.

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160[K].—C. K. TSAO, "Distribution of the sum in random samples from a discrete population, "*Ann. Math. Stat.*, v. 27, 1956, p. 703–712.

Let $X_1, X_2, \cdots X_m$ be a random sample drawn from the population having the discrete density,

$$f(x; p) = p_x ; \qquad\qquad x = 1, 2, \cdots k;$$

where $p = (p_1, p_2, \cdots p_k)$, $\sum p_i = 1$, and let $S = \sum_{i=1}^{n} X_i$. This paper is concerned with the distribution of S. For $k = 2$, the distribution of S is binomial. For $K > 2$, the *pdf* of s, denoted by $g(s, p, m)$ can be obtained from a generating function displayed in the paper.

In particular, the author gives tables for the distribution in the special case for $p_i = p_j = 1/k$ for $k = 3, 4, 5, 6$ and $m = 1$, (1) 20 to at least $5S$.

Normal approximations are investigated and some applications are briefly discussed.

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161[K].—J. W. Tukey, "Keeping moment-like sampling computations simple," *Ann. Math. Stat.*, v. 29, 1956, p. 37–54.

The author provides a table to order 8 for expressing generalized k-statistics (for which he proposes the name "polykays") and augmented symmetric functions ("brackets") in terms of each other. This table is more convenient, over its range, than the table to order 12 of Abdel-Aty [1].

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1. S. H. Abdel-Aty, "Tables of generalized k-statistics," *Biometrika*, v. 41, 1954, p. 253–260. [*MTAC*, v. 9 *RMT* **9**, 1955, p. 30].

162[K].—J. E. Walsh, "Asymptotic efficiencies of a nonparametric life test for smaller percentiles of a gamma distribution," Amer. Stat. Assn., *Jn.*, v. 51, 1956, p. 467–480.

The author assumes that the time at failure, t, in a life test, a non-negative variate is subject to the gamma density function

$$f(t, \xi, \lambda) = \xi^{-\lambda} t^{\lambda-1} e^{-t/\xi} / \Gamma(\lambda)$$

with positive parameters ξ and λ. For $\xi = 2$ and λ and integral multiple of $\frac{1}{2}$, a Chi-square distribution is obtained. If λ increases the distribution tends to normal with mean $\lambda\xi$ and variance $\lambda\xi^2$. The number of items tested simultaneously is n. Let F be the reciprocal of the lower bound efficiency value E. In determining asymptotic efficiencies it is assumed that λ is known. This decreases the efficiency of non parametric tests. If only a fraction of the sample failed and the distribution is normal ($\lambda \to \infty$) Table 1 gives the asymptotic ($\eta \to \infty$) efficiency lower bound, and F to 2D for the population percentage points $q = .01, .02, .05, .1, .2, .3, .5, .7$. With increasing q the efficiency decreases from .99 to .61.

For the same q and known values of the parameter $\lambda = .05, .1, .5, 1, 2, 5, 10, 20, 50$ table 2 yields in the case that all n items fail the value of F to $\geq 2S$ so that non parametric results based on a sample of nF items furnish at least as much information as the corresponding best results based on all values n. For fixed q and increasing λ the values nF decrease for example for $q = .01$ the value F decreases from 1980 valid for $\lambda = .05$ to 18.7 valid for $\lambda = 50$. However for $q = .7$ the value F

decreases first and increases slightly later. For fixed λ the value F decreases with q increasing. The procedure leads to a considerable increase in sample size varying from 1980 n for $\lambda = .05$, $q = .01$ to 1.83 n valid for $\lambda = 50$, $q = .7$.

Table 3 gives values of $\phi_9(\lambda)$, the 100 q probability point of the standardized variate t/ξ, for the same values of λ, the previous values of q and $q = .9, .95, .99, .999, .9999, .99999$ to $3S$ in nearly all cases. The values $\phi_9(\lambda)$ vary from 6.10^{-41} valid for $\lambda = .05$, $q = .01$ to 86 valid for $\lambda = 50$ and $q = 1 - 10^{-5}$.

No indication is given how the tables should be interpolated. The paper does not make easy reading.

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163[K].—BARNET WOOLF, "The log likelihood ratio test (the g-test). Methods and tables for tests of heterogeneity in contingency tables," *Ann. Human Genetics*, v. 21, 1957, p. 397–409.

The purpose of these tables is to facilitate the computation of the log likelihood ratio statistic for tests of heterogeneity in contingency tables. The test statistic G is defined as $G = 2 \sum a(\ln a - \ln m)$, where a and m refer to observed and expected cell frequencies, respectively, and the summation is over all cells. Tables 1 and 2 give $g(a) = 2 a \ln a$ to 4D for $a = 1(1)$ 2009 and $a = \frac{1}{2}(1)$ 290 $\frac{1}{2}$. Interpolation in table 1 is discussed. An approximation to $g(x)$ for large x is given. Some of the terms in the approximation involve $2/x$ (table 3) given to 4D for x in the range 297 to 2105, $r(p - r)/(2p)$ (table 4) given as rational fractions for $p = 2(1)$ 10, $r = 1(1)$ 5, $p > r$, and $2 \ln p$ (table 5) given to 9D for $p = 2(1)$ 11, 13, 17, 19, 20, 40, 50, 100.

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164[K].—B. F. KIMBALL, "The bias in certain estimates of the parameters of the extreme-value distribution," *Ann. Math. Stat.*, v. 27, 1956, p. 758–767.

The paper deals almost entirely with computing the bias of a new (and simple to compute) estimate of the reciprocal of the scale parameter α in the Type I extreme value distribution described by the c.d.f.

$$\phi(x, \alpha, u) = \exp\{- \exp[-\alpha(x - u)]\}.$$

The new estimate of α is suggested by its maximum likelihood estimate (which is difficult to compute explicitly). For a sample of size n, where the observations are ordered: $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, the estimate $\hat{\beta}$ of $1/\alpha$ is given by $\hat{\beta} = \sum_{i=1} c_j x_j$, where $c_j = (1 - \sum_{i=j}^{n} 1/i)n^{-1}$. Formulae for the constants b_n which are such that $E(b_n\hat{\beta}) = 1/\alpha$ are worked out for any size sample n and tabulated to 4D for $n = 2(1)112$. The question of estimating the parameter u, when α is known, is treated briefly. Formulae are given for the bias and variance of this estimator. Results for Type I extreme value distributions can be extended readily to those of Types II and III. The author mentions that computation of the bias factors, b_n, was facili-

tated by the apparently unpublished table of $\Delta^i \log 1$ from $i = 1$ to $i = 111$, which has been prepared by the National Bureau of Standards.

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165[K].—F. G. FOSTER, & D. H. REES, "Upper percentage points of the generalized beta distribution. I.," *Biometrika*, v. 44, 1957, p. 237–247.

Let θ_{max} denote the greatest root of $[v_2 B - \theta(v_1 A = v_2 B)] = 0$ where A and B are independent estimates, based on v_1 and v_2 degrees of freedom, of a parent dispersion matrix of a bivariate normal population. The authors find the $100P\%$ points to 4D of the distribution

$$I_x (2; p, q) = Pr\{\theta_{max} \leq X\},$$

where $I_x (2; p, q)$ is the distribution function, of the greatest root, $P = .80$ (.05) .95, .99, $v_1 = 5(2)41(10)121, 161, v_2 = 2, 3(2)21, v_1 = 2q + 1, v_2 = 2p + 1$.

Examples illustrate the use of the tables.

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166[K].—P. V. K. IYER & M. N. BHATTACHARYYA, "On some statistics comparing two binomial sequences," Indian Soc. Agricultural Stat., *Jn.*, v. 7, 1955, p. 187–213.

Let $x_1, x_2, \cdots x_n$ and $y_1, y_2, \cdots, y_n$ be two observed sequences in the order of drawing from binomial universes in which each x_i takes the value A or B with probabilities p_x and $1 - p_x$ and each y_i takes the same values with probabilities p_y and $1 - p_y$. Let $(x - y) = (A - B), (B - A), (A - A), (B - B)$ be assigned the values $1, -1, 0, 0$, respectively and $[x - y] = [A - B], [B - A], [A - A],$ $[B - B]$ be assigned the values $1, 1, 0, 0$ respectively. In the main portion of the paper the authors study the moment characteristics of two systems of statistics they propose for testing the hypothesis that $p_x = p_y = p$. A special case of the one system is $X_0 = \sum_{r=1}^{n} (x_r - y_r)$ and a special case of the other is

$$Y_1 = \sum_{r=1}^{n} [x_r - y_r] + \sum_{r=1}^{n-1} \{[x_r - y_{r+1}] + [x_{r+1} - y_r]\}.$$

The exact probability distributions of these two quantities are tabled for $p = .1(.1).5$ and $n = 1(1)6$ for x_0 and $n = 2(1)6$ for Y_1 to 4 or more D.

C.C.C.

167[K].—F. G. FOSTER, "Upper percentage points of the generalized beta distribution. II," *Biometrika*, v. 44, 1957, p. 441–453.

Let θ_{max} denote the greatest root of $[v_2 B - (v_1 A + v_2 B)] = 0$, where A and B are independent estimates, based on v_1 and v_2 degrees of freedom, of a parent dispersion matrix of a trivariate normal population. Define $I_x(3; p, q) = Pr\{\theta_{max} \leq x\}$, where $p = \frac{1}{2}(v_2 - 2)$, $q = \frac{1}{2}(v_1 - 2)$. The author tabulates the 80, 85, 90, 95, and 99%

points of I_x $(3; p, q)$ to 4D for $v_1 = 4(2)194$, $v_2 = 3(1)10$. Previously I_z $(2; p, q)$ had been tabulated [1].

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1. F. G. FOSTER & D. H. REES, "Upper percentage points of the generalized Beta distribution. I," *Biometrika*, v. 44, 1957, p. 237–247. [MTAC; this issue]

168[K].—D. E. BARTON & F. N. DAVID, "Multiple runs," *Biometrika*, v. 44, 1957, p. 168–178.

The authors are interested in the number of groups (runs), T, formed from r items of k different colors, $r = \sum_{i=1}^{k} r_i$. The values of T and $\sum T$, from which the probability of obtaining T may be calculated, are explicitly stated for $k = 3$ and 4, $r = 6(1) 12$ in Tables 1 and 2. The first four factorial moments and factorial cumulants of $S = r - T$ are derived. For $r > 12$ it is suggested that the normal approximation is adequate for tests of significance. Applications are made to the analysis of variance.

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169[L].—NBS Applied Mathematics Series, No. 52, *Integrals of Airy Functions*, U. S. Government Printing Office, Washington, D. C., 1958, iii + 28 p., 26 cm. Price $.25.

This publication falls into two halves, pages 1–14 being devoted to "Tables of integrals of the Airy function $Ai(-x)$," with introduction by E. E. Osborne, and pages 15–28 to "Tables of the modified Airy integral $A_0(x)$," with introduction by P. Rabinowitz.

In the first part are tabulated the two functions

$$f(x) = \int_0^x Ai(-t)\, dt, \qquad F(x) = \int_0^x f(t)\, dt$$

where

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3} t^3 + xt\right) dt.$$

The tables give $f(x)$ to 8D and $F(x)$ to 7D, both for $x = -2(.01) + 5$ and with second differences throughout.

In the second part the main table relates to

$$A_0(x) = \int_0^\infty \exp\left(-\frac{1}{3} t^3 - xt\right) dt$$

Both $A_0(x)$ and its negative derivative $-A_0'(x)$ are tabulated to 8D, for $x = 0(.01)1(.02)5(.05)11$ with second differences, and for $1/x = .01(.01).10$ with modified second differences. There is also a small table giving

$$G(x) = \int_0^x A_0(t)\, dt$$

to 8D without differences for $x = .5, 1(1)11$.

The authors give references to several other recent tables of the same or almost the same functions, but in each case of overlap the NBS tables have either an extra decimal or a smaller interval (usually both) compared with the other published tables, so that the NBS tables make a welcome contribution.

A.F.

170[L].—K. A. KARPOV, *Tablicy Funkciĭ* $w(z) = e^{-z^2} \int_0^z e^{z^2} dx$ *v kompleksnoĭ oblasti,*

Insdat. Akad. Nauk SSSR, Moscow, 1954, 536 p., 21 × 27 cm., 1 insert. Price 61 rubles.

This volume contains 5D tables of the real and imaginary parts of the function

$$w(z) = e^{-z^2} \int_0^z e^{z^2} dx = u + iv$$

for $x = \rho e^{i\theta}$, $0 \leq \rho \leq 5$, $0 \leq \theta \leq \pi/4$ and $\theta = \pi/2$. For $\theta = 0$, ρ runs to 10. The intervals in ρ and θ vary. In the introduction a diagram is given representing the values of θ included in the volume and the range of ρ for each θ. A table indicates the intervals in ρ in various parts of the volume. The diagram (but not the table) is reproduced also on a cardboard insert which serves as an index to the numerical tables.

Using the symmetry properties of $w(z)$, this function can be evaluated by means of the present tables on the imaginary axis and in a sector of half-angle 45° to both sides of the real axis. Thus, one half of the complex plane remains uncharted except for the imaginary axis.

In spite of this uncharted part, this is the most extensive table known to exist of the error integral in the complex plane. The only other source of adequate numerical material is due to Clemmow and Munford [1] whose table gives the real and imaginary parts of

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{\frac{1}{2}i\pi\rho^2} \int_\rho^\infty e^{-\frac{1}{2}i\pi t^2} dt$$

for $| \rho | = 0(.01).8$ and $\arg \rho = 0(1°)45°$, thus covering a part of the region conjugate complex to that covered in the present tables. Graphs of curves of constant modulus and argument of $\int_0^z e^{-z^2} dx$ were earlier given by Laible [2] in a somewhat larger portion of the first quadrant, in particular in a part of the sector uncharted by the numerical tables.

The introduction gives power series expansions and asymptotic expansions for the functions tabulated here, graphs of u and v qua functions of ρ (for selected values of θ), relief diagrams of u and v over the sector of tabulation, a description of the tables and numerical examples showing their use, and two one page tables of coefficients in the asymptotic expansions.

It may be mentioned that the error function with complex variable occurs in certain wave propagation problems and the curious concentration on one half of

the complex plane is motivated by the values of the complex variable occurring in those problems.

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1. P. C. CLEMMOW & CLARA M. MUNFORD, "A table of $\sqrt{(\frac{1}{2}\pi)}\, e^{\frac{1}{2}\pi i \rho^2} \int_\rho^\infty e^{-\frac{1}{2}\pi i \lambda^2}\, d\lambda$ for complex values of ρ," Roy. Soc. *Phil. Trans.*, Series A, No. 895, v. 245, 1952, p. 189–211.
2. THEODOR LAIBLE, "Hohenkarte des Fehlerintegrals," *Zeit. Angew. Math. Phys.*, v. 2, 1951, p. 484–486.

171[L].—B. LOHMANDER & S. RITTSTEN, *Table of the function* $y = e^{-x^2}\int_0^x e^{t^2}\, dt$,

Lund University, Department of Numerical Analysis, Table No. 4. (From *K. Fysiogr. Sällsk. i Lund Förhandlingar*, v. 28, No. 6, 1958, p. 45–52.) C. W. K. Gleerup, Lund, Sweden, 1958, 24.2 cm.

This basic table was prepared on SMIL, the electronic computer of Lund University. Tables 1 and 2 give y to 10D with modified second differences for $x = 0(.01)3$ $(.02)5$ and $1/x = 0(.005).200$ respectively. Tables 3 gives y to 20D without differences for $x = .5(.5)10$. Table 4 gives particulars of the maximum of the function and the point of inflexion.

The only previous table to 10D is that of Rosser [1], in which the authors find only two rounding errors.

A.F.

1. J. B. ROSSER, Theory and Application of $\int_0^x e^{-x^2}\, dx$, etc., limited circulation 1945, publication in book form Brooklyn, N. Y., 1948. [See *MTAC*, v. 2, 1947, RMT 351, p. 213, and v. 3, 1949, RMT 642, p. 474.]

172[L].—ELLEN BRAUER & JANE C. GAGER, "Table of the error function for complex arguments," 92 sheets, 28 x 16 cm. Deposited in the UMT File.

This table gives the real and imaginary parts of

$$\int_0^z e^{t^2}\, dt$$

for $z = re^{i\theta}$ where $r = .1(.1)2$ and $1.51(.1)1.59$ for $\theta = 0(1°)90°$. It was prepared on the IBM 704 at the National Bureau of Standards by summing an appropriate number of terms of the power series. The results are given to 9D, 7 of which are thought to be accurate.

The table was spot-checked by comparison with the tables of Karpov and Faddeeva and Terentiev and with manuscript tables by F. J. Stockmal and W. F. Cahill of the National Bureau of Standards.

The table was prepared in connection with an investigation of the radius of univalence of erf z.

ELLEN BRAUER
JANE C. GAGER

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1. K. A. KARPOV, *Tablicy Funkcii* $w(z) = e^{-z^2} \int_0^z e^{z^2} dx$ *v kompleksnoioblasti* (Izdat. Akad.

Nauk SSSR, Moscow, 1954.) [*MTAC*, Rev. 170, v. 12, 1958, p. 304].
2. V. N. FADDEEVA & N. M. TERENTIEV, *Tablicy znacenii funckii*

$$w(z) = e^{-z^2} \left(1 + (2i/\sqrt{\pi}) \int_0^z e^{t^2} dt\right)$$

ot kompleksnogo argumenta, Gosudarstv. Izdat. Tekn.-Teor. Lit., Moscow, 1954.
3. E. KREYSIG & JOHN TODD, "Numerische Mathematik," Amer. Math. Soc., *Bull.*, to be published.

173[M].—CHIH-BING LING, "Tables of values of 16 integrals of algebraic-hyperbolic type," *MTAC*, v. 11, 1957, p. 160–166.

The integrals are

$$\int_0^\infty f(k, x)k!(\sinh 2x \pm \sin 2x)^{-1} dx$$

where $f(k, x)$ has the following values $(2x)^k$, $(2x)^k e^{-2x}$, $(2x)^k \tanh x$, $(2x)^k \coth x$, $x^k \sinh x$, $x^k \cosh x$, $x^k \tanh x \sinh x$, $x^k \coth x \cosh x$, for the integral values of k for which they are convergent. The limiting value, as $k \to \infty$, for all except the second is unity; the asymptotic value of the second is $2^{-(k+1)}$. Values are given to 6D, and tabulation is continued until the limiting values are attained, which happens when k is between 15 and 25. The two integrals for $f(k, x) = (2x)^k$, and the two for $f(k, x) = (2x)^k l^{-2x}$ were studied by R. C. J. Howland and tabulated by him [1] and by Howland and A. C. Stevenson [2]; further tabulations were made by the present author and C. W. Nelson [3]. All the others can be expressed in terms of these and the Glaisher Series $1 \pm 2^{-k} + 3^{-k} \pm 4^{-k} + \cdots$, $1 \pm 3^{-k} + 5^{-k} \pm 7^k \cdots$. Various functional relations between the sixteen integrals were used for checking purposes.

These integrals occur in elastic problems concerning perforated or notched strips.

JOHN TODD

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1. R. C. J. HOWLAND, Royal Soc., *Phil. Trans.*, v. 229 A, 1930, 49–86.
2. R. C. J. HOWLAND & A. C. STEVENSON, Royal Soc., *Phil. Trans.*, v. 232 A, 1936, p. 155–222.
3. CHIH-BING LING & C. W. NELSON, *Ann. Acad. Sci.*, Taiwan, China, v. 2, 1955, p. 45–50.

174[S, X].—E. U. CONDON & HUGH ODISHAW, Editors, *Handbook of Physics*, McGraw-Hill, New York, 1958, xxvi + 1472 p., 24.8 cm. Price $25.00

No critical review of this volume will be attempted here. The book was prepared under Condon's guidance as the one book a physicist marooned on a desert island would find most useful. Many chapters are written by Condon; others are by authorities in their fields.

The parts include Mathematics, Mechanics of Particles and Rigid Bodies, Mechanics of Deformable Bodies, Electricity and Magnetism, Heat and Thermodynamics, Optics, Atomic Physics, The Solid State, and Nuclear Physics. Each part contains several chapters expounding the present state of the science.

There is only incidental material on geophysics or astrophysics. Tables appear only incidentally. Some attention is paid to computation, particularly in the part on mathematics. There is a chapter devoted to Units and Conversion Factors.

There is a table of contents at the beginning of each part, but the reader must find the right part himself, for there is no master table of contents. The index is incomplete; for example, there is no reference under Avogadro's number to the material furnished by Jesse W. M. Dumond and E. Richard Cohen on "Fundamental Constants of Atomic Physics," although this is one of the primary constants. Numerous other omissions came to this reader's attention.

Again, without attempting to be critical, it is hard for the reviewer (who contributed in a minor way to the volume) to suggest a place where more physics is contained in a single volume.

<div style="text-align: right">C. B. T.</div>

175[S, X, Z].—M. Lotkin & C. Berndtson, "Least Squares polynomial fits for the ICAO (1956) atmospheric density," RAD Technical Memo, RAD-2-TM-58-59, 1958, Research and Advanced Development Division, AVCO Manufacturing Corporation, Lawrence, Mass. 19 p., $8\frac{1}{2}''$ x $11''$. Available on request from AVCO Corp.

Thirteen Polynomial approximations to the density (in gm^{-3} grams per cubic meter) using polynomials varying from fourth to sixth degree. Accuracy of about $4S$ is claimed for most regions. The independent variable is geopotential altitude, and the atmosphere used is the U. S. Standard Atmosphere [1].

A table is appended to give density ρ as a function of geopotential altitude H, $5S$, for $H = 0(1)130(5)300(10)500$ km.

The polynomial approximations were prepared to be useful in calculations involving automatic computers. No easy means of obtaining less accurate lower degree polynomials (for use with analogue computers) other than by truncation are given. However the user might consult [2] for this.

<div style="text-align: right">C.B.T.</div>

1. R. A. Minzer & W. S. Ripley, "The ARDC Model Atmosphere," 1956, Air Force Surveys in Geophysics No. 86, Air Force Cambridge Research Center, Dec. 1956.
2. C. Lanczos, *Tables of Chebyshev Polynomials*, $S_n(x)$ and $C_n(x)$, NBS Applied Mathematics Series, No. 9, 1952, Washington.

176[V, Z, P].—Proceedings of the Symposium on Digital Computing in the Aircraft Industry, Jan. 31–Feb. 1, 1957, IBM-NYU, 1957, 28 cm. Price $8.00. Available from Max Woodbury, New York University, Research Division, 401 West 205 Street, New York, New York. Paper bound.

This volume contains the texts of most of the papers presented at a symposium on computing in the aircraft industry, held on January 31 and February 1, 1957 at New York University under the joint sponsorship of I.B.M. and N.Y.U.

The twenty-three papers are grouped into four categories bearing the imposing titles: "The Aircraft and Its Macrocosm"; "The Computer and Its Macrocosm"; "The Computer and The Aircraft"; and "The Computer and Its Mathematics." The apprehensive reader, comfortably ensconsed in his own microcosm, will find

the material in most of the papers considerably less grandiose than the above titles would indicate.

With very few exceptions, the emphasis throughout the symposium is on what may be termed classical aeronautical engineering as opposed to the modern problems associated with space technology. Thus, the six papers in the first category deal with rather conventional aeronautical engineering problems. Of these, two are worthy of mention. The paper by T. F. Cartaino and S. E. Dreyfus presents a different approach to the variational problem of determining the minimum time-to-climb for an airplane. The authors illustrate how dynamic programming is used to solve this problem. Another interesting paper, by U. O. Lappi, compares two mathematical tools now very much in vogue for the frequency analysis of data: autocorrelation analysis, and the method of transfer functions for systems governed by constant-coefficient linear differential equations. These two techniques are applied to the problem of determining the spectra of gust loads on airplanes, and results are compared.

In the second category, two of the papers discuss specific installations where digital computers are used in on-line data reduction. Although the systems described are no lonter new, some of the details may prove useful to those interested in this particular aspect of digital computation.

In the third category, the paper by C. Eisen and T. Rivlin gives some specific results on a procedure for computing conformal mappings, and applying this to the study of flows in various channels. Most of the other papers are brief and expository.

The last category contains most of the meat of the symposium. Topics treated include computation of missile trajectories, generation of sampled filtered noise on a digital computer, numerical solution of ordinary differential equations, and eigenvalue problems.

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177[W].—L. G. Peck & R. N. Hazelwood, *Finite Queuing Tables*, Wiley, New York, 1958, xvi + 210 p., 28 cm. Price $8.50

The meat of this book is a single long table, which lists steady-state values of the probability, D, that a random customer will be delayed, and of the efficiency factor.

$F = 1 -$ (Average proportion of population waiting for service), for finite-population queuing models in which both service time and "idle time" (between completion of a customer's service and his next service need) are exponentially distributed. These quantities are listed to three decimal places, as functions of the number of customers in the population serviced, N, the ratio

$$X = \frac{\text{Average service time}}{\text{Average service time, plus average idle time}},$$

and the number of servers, M. The values of N run from 4 to 250 and those of X from 0.001 to 0.950, by increments as follows:

N	ΔN	X	ΔX
4–26	1	0.001–0.026	0.001
26–70	2	0.026–0.070	0.002
70–170	5	0.070–0.170	0.005
170–250	10	0.170–0.340	0.010
		0.340–0.600	0.020
		0.600–0.950	0.050

The varying limits on M are such that the most interesting cases are covered. Conservative programming guarded against roundoff errors, and mechanical and photographic preparation of plates directly from UNIVAC I output tapes eliminated manual transcription and typesetting.

A brief preface outlines the underlying theory and the computing technique used, and explains the use of the tables. Simple expressions are given for computing, from knowledge of F, the average proportions of the population being serviced, "idle," and waiting for service. (A slight oddity, to the reviewer, is the description of a "productive" customer, i.e., one neither being serviced nor waiting for service, as "idle.") Two typical applications are described and worked out numerically. The only proof error discovered which is worthy of note is at the top of page ix: the first line could correctly read "where $\xi = \lambda T$;" and the left side of the following equation should be ξ, not T.

This volume is undoubtedly a valuable contribution from the point of view of operations researchers, who find frequent use for finite queuing models. Perhaps more important, its publication is a long step toward the use of queuing theory as an everyday tool of industrial engineering. It is to be hoped that unsophisticated users will not, because of the availability of these tables, be overwilling to pretend that every distribution in real-life queuing problems is exponential.

<div align="right">JAMES R. JACKSON</div>

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178[X].—NBS Applied Mathematics Series, No. 49, *Further Contributions to the Solution of Simultaneous Linear Equations and the Determination of Eigenvalues,* U. S. Government Printing Office, Washington, D. C., 1958, iv + 81 p., 26 cm. Price $.50

This pamphlet consists of three papers: "Kernel polynomials in linear algebra and their numerical applications," by E. L. Stiefel, "The quotient-difference algorithm," by P. Henrici, and "Solution of eigenvalue problems with the LR-transformation," by H. Rutishauser.

In the first paper, the theory of orthogonal polynomials is shown to provide a common theoretical background for several modern iteration techniques. A multilithed version of this paper was previously reviewed in this journal [1].

The quotient-difference algorithm was announced by Stiefel in 1953 and developed by Rutishauser in a series of papers [2]. The article on the algorithm by Henrici appears to be the only self-contained exposition in English. The discussion is based on classical theorems of analytic functions, whereas Rutishauser's treat-

ment uses special properties of continued fractions. Basically, the determination of the eigenvalues of a matrix A is equivalent to the determination of the poles of the rational function

$$f(z) = x^T(I - zA)^{-1}y = \sum_{k=0}^{\infty} s_k z^k$$

where x and y are (almost) arbitrary vectors and s_k the Schwarz constants

$$s_k = x^T A^k y.$$

The derivation of the quotient-difference algorithm thus is made to follow from Hadamard's study of the singularities of a function given by the Taylor series at the origin.

The quotient-difference algorithm consists of a two-dimensional array of quantities, $q_k^{(n)}$, $e_k^{(n)}$, associated with a rational function $f(z)$ (more generally a function meromorphic in a region $|z| < R$), satisfying the rhombus rules:

$$q_{k+1}^{(n)} \cdot e_k^{(n)} = q_k^{(n+1)} \cdot e_k^{(n+1)},$$
$$q_k^{(n)} + e_k^{(n)} = q_k^{(n+1)} + e_{k+1}^{(n+1)}$$
$$(k = 1, 2, \cdots; n = 0, 1, 2, \cdots).$$

The quantities $e_0^{(n)} = 0$ for all values of n. The first column $q_1^{(n)}$ ($n = 0, 1, 2, \cdots$) or a diagonal

$$e_0^{(0)}, q_1^{(0)}, e_1^{(0)}, \cdots, q_k^{(0)}, e_k^{(0)}, \cdots$$

may be determined from the coefficients s_k. If the function $f(z)$ satisfies suitable conditions, and the algorithm can be carried out,

$$\lim_{n \to \infty} q_k^{(n)} = \lambda_k,$$

where λ_k is the kth pole of $f(z)$ in order of decreasing modulus. The method using the first column as a starting point is numerically unstable. If the algorithm is begun with a diagonal, the process is apparently stable, though this has not been proved. An accelerated form of the algorithm may be used in some cases.

Applications of the quotient-difference algorithm include the determination of the eigenvalues and of the eigenvectors of a matrix, of the zeros of a polynomial, and of the continued fraction expansion corresponding to a function given by a Taylor series. The algorithm is most conveniently used for finding matrix eigenvalues if the matrix first is reduced to Jacobi form.

The LR-transformation discussed in the third paper is an iterative process for finding matrix eigenvalues and vectors which for Jacobi matrices reduces to the quotient difference algorithm. Given any matrix $A = A_0$, sequences of matrices, L_k, R_k, Λ_k, may be determined so that

$$A_{k-1} = L_k R_k$$
$$R_k L_k = A_k \qquad (k = 1, 2, \cdots)$$
$$\Lambda_k = L_1 L_2 \cdots L_k$$

where L_k is lower triangular with ones on the main diagonal and R_k upper triangular. Let also

$$\Lambda_\infty = \lim_{k \to \infty} \Lambda_k \,,$$

$$A_\infty = \lim_{k \to \infty} A_k \,.$$

If Λ_∞ exists, so does A_∞, and $\Lambda_\infty A_\infty = A \Lambda_\infty$; then A_∞ is an upper triangular matrix whose diagonal elements are the eigenvalues of A. The convergence of A_k is assured if the matrix is hermitian and positive definite; it is also assured if certain less restrictive conditions are satisfied.

The numerical examples discussed include Wilson's matrix, a matrix with repeated eigenvalues, and a matrix with "disorder of latent roots." Techniques for improving the convergence in the case of real matrices with either real or pairs of conjugate imaginary roots are developed. The paper concludes with a continuous analogue and a Graeffe-like modification of the LR-transformation.

<div align="right">A. A. GRAU</div>

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1. EDUARD L. STIEFEL, *Kernel polynomials in linear algebra and their numerical applications*, multilithed from typescript, 1955. [*MTAC*, Rev. 87, v. 9, 1955, p. 199–200.]
2. See the bibliography in Henrici's paper.

179[X, I].—D. J. PANOW, *Formelsammlung zur numerischen Behandlung partieller Differentialgleichungen nach dem Differenzenverfahren*, Akademie-Verlag, Berlin, 1955, x + 134 p., 24 cm, Price DM 12,—.

This is a translation into German by Karl Borkmann and Werner Schulz of the fifth Russian edition [6] of a book whose first edition was issued in 1938 with 129 p.

The book is a handbook of techniques, intended to tell the human computer, armed with pencil, paper, and perhaps a keyboard calculator, how to set up and solve difference equations corresponding to certain comparatively simple partial differential equations. Most attention is given to the Dirichlet, Neumann, and third boundary-value problems for Poisson's equation, but there are sections on several other types of problems. No attention whatever is given to roundoff error, which is presumably negligible with a desk calculator, and what little is said about the discretization error (the difference between the solution of the difference equations and that of the differential equations) is sadly misleading.

The material is presented with remarkable clarity, thanks largely to 114 line drawings and tables in 128 pages of text. Almost every difference expression is given an eye-catching presentation in a box, with a sketch of the network and clearly numbered points, the relevant formula, and an unexplained indication of the order of the error. Each method of solution is illustrated with one or more examples, followed by tables of as many as 40 successive approximations to the solution. With this book at hand a reader of German should be able to approximate and solve any difference equation of the types considered. But the reader will have learned nothing whatever of the theory behind what he is doing.

The reviewer knows of no comparable handbook. Somewhat related are textbooks by Milne [5] and by Crandall [1], which treat the solution of partial differential equations by difference methods, and also give numerous examples and drawings. But as textbooks they have the much larger purpose of giving the reader some theoretical insight into the problems.

Section 1 is an introduction to difference methods via the Dirichlet problem for Laplace's equation. Section 2 introduces tabular differences for one variable and gives formulas and tables for interpolation and finding first derivatives from differences, as summarized below. More formulas give up to the 4th derivative at tabular points. There are three pages of simple formulas for $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, $\partial^2 u/\partial x\partial y$, $\partial^4 u/\partial x^4$, $\partial^4 u/\partial y^4$, and $\partial^4 u/\partial x^2 \partial y^2$ at a node of a square grid. For $\partial^4 u/\partial x^4$, for example, there are formulas for three points, for five points, and for nine points. At the end of the section is an explanation of "extrapolation to zero mesh size," here called "Runge's principle." It is stated flatly that in solving the Poisson equation in a square net with Collatz's interpolation method at the boundary, the discretization error has the order of magnitude h^2. This assertion has been proved only under stringent hypotheses which cannot often be expected to hold. Whenever a region has a reëntrant corner, for example, we can expect Panow's assertion to be false, even for smooth boundary values. See Laasonen [3] for some experimental evidence.

Section 3 is devoted to the Laplace and Poisson equations in 2 dimensions—setting up the difference equations with details on boundary interpolation according to Collatz and Mikeladze, and suggestions on solving the difference equations. Gradual refinement of the net is recommended, together with use of Runge's principle and an idea of Liūsternik for accelerating the convergence to the solution of the difference equations for a fixed mesh size. The Neumann and third boundary-value problems are discussed, and the reader is warned that the approximation of the normal derivative at the boundary C is not well understood except when C is a vertical or horizontal line through net points. For the Neumann and third boundary-value problems the author recommends guessing boundary values, solving the resulting Dirichlet problem, then correcting the boundary values, etc. As one of the simpler illustrative examples, the author solves Laplace's equation in a rectangle, R, with fixed boundary values on one side of C (the boundary of R), with a homogeneous third boundary condition on part of another side of C, and with a zero normal derivative elsewhere on C. We are given 10 different approximate solutions.

The later sections are shorter and less thorough. In Section 4 the biharmonic equation is treated, mostly by reducing it to a pair of Poisson equations. The heat equation is treated in Section 5 for two space dimensions and possibly curved boundaries. (Since stability and round-off are not treated, the ratio between the space mesh length and the time difference is left arbitrary for the usual formulas!) Section 6 treats the wave and telegrapher's equation with fixed networks. A fairly long Section 7 concludes the book by discussing quasilinear hyperbolic systems, mainly by Massau's method of characteristics.

There are 38 references, reasonably balanced between Russian and western work, but the latest reference is 1948. There is a list of the illustrative examples, but no index. The printing and paper are excellent.

ADDENDUM

The tables of interpolation coefficients are not mentioned by Fletcher, Miller, and Rosenhead [2], nor by MTAC to date, and are only incompletely listed by Lebedev and Fedorova [4]. For the sake of the record, they are recorded here in approximately the notation of [2]:

3-point Lagrange interpolation coefficients: 5D, $\pm$ $n = 0(.01).5$.

4-point Lagrange interpolation coefficients: 5D, $n = 0(.01)1$.

Gregory-Newton interpolation coefficients: 5D, 2nd; and 4D, 3rd and 4th; $n = 0(.01)1$.

Stirling interpolation coefficients: 5D, 2nd; and 4D, 3rd and 4th; $n = 0(.01).5$.

Bessel interpolation coefficients for odd and mean even differences: 5D, 2nd; and 4D, 3rd and 4th; $n = 0(.01)1$.

First derivatives from Stirling differences: 5D, 3rd; and 4D, 4th–6th; $n = \pm 0(.01).25$.

First derivatives from Bessel differences: 5D, 3rd; and 4D, 4th–6th; $n = .5(.01).75$.

No differences are tabulated, and no sources are given for the tables.

GEORGE E. FORSYTHE

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1. S. H. CRANDALL, *Engineering Analysis, a Survey of Numerical Procedures*, McGraw-Hill Book Co., Inc., New York, 1956 [Rev. **99**, *MTAC*, v. 11, p. 219.]
2. A. FLETCHER, J. C. P. MILLER, & L. ROSENHEAD, *An Index of Mathematical Tables*, McGraw-Hill Book Co., Inc., New York, 1946.
3. PENTTI LAASONEN, "On the truncation error of discrete approximations to the solutions of Dirichlet problems in a domain with corners," Assoc. Comput. Mach., *Jn.*, v. 5, 1958, p. 32–38.
4. A. V. LEBEDEV & R. M. FEDOROVA, *Spravochnik po Matematicheskim Tablitsam*, Izdatel'stvo Akademii Nauk SSSR, Moskva, 1956.
5. WILLIAM EDMUND MILNE, *Numerical Solution of Differential Equations*, John Wiley and Sons, Inc., New York, 1953.
6. D. IU. PANOV, *Spravochnik po Chislennomu Resheniiu Differentsial'nykh Uravnenii v Chastnykh Proizvodnykh*, 5th ed., Gostekhizdat, Moscow-Leningrad, 1951.

180[X, P, S].—I. SOKOLNIKOFF & R. REDHEFFER, *Mathematics of Physics and Modern Engineering*, McGraw-Hill Book Co., New York, 1958, ix + 810 p., 22.6 cm. Price $9.50

TABLE OF CONTENTS

"In the sense that a working course in calculus is the sole technical prerequisite, this book is suitable for the beginner in applied mathematics. ... When taken in

sequence, this book has ample substance for four consecutive semester courses meeting three hours a week."

The chapter on Numerical Analysis is devoted to a simple discussion of (a) methods for solving equations (roots, solution of linear systems); (b) polynomial interpolation (with and without differences), least squares fitting and a short section on trigonometric interpolation; (c) numerical integration of differential equations with a short section on the numerical solution of partial differential equation problems (previously discussed in the chapter on Partial Differential Equations). The emphasis is placed on careful explanation of methods including numerical examples. The maximum principle for the solution of difference equations corresponding to the potential equation is proved in a paragraph of chapter 6 on Partial Differential Equation. The choice of mesh width ratios for the wave equation and the equation of heat conduction is not considered.

The one page table of $\Phi(x)$ is reproduced from *Biometrica Tables for Statisticians*, v. 1, 1954, edited by E. S. Pearson and H. O. Hartley, Cambridge University Press.

$$0.00 \ (.01) \ 3.09 \ (4 \ \text{decimals}); \ 1.0(.2)4.8 \ (7 \ \text{decimals}).$$

E.I.

181[X, F].—P. B. FISCHER, *Arithmetik*, Third Edition, Walter de Gruyter and Co., Berlin, 1958, 152 p., 15.5 cm. Price DM 2,40, paper bound.

A review of elementary arithmetic derived fairly rigorously from an assumed sequence of natural numbers. There is a brief but excellent running historical commentary on the concepts, symbols and methods. Of special interest is the section on root extraction. A number of special results, including the use of continued fractions, which are not normally included in elementary courses but which are of use to computers are given in this section. There is a brief mention of complex numbers, quaternions, and some extremely elementary combinatorics.

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182[Z].—MAURICE V. WILKES, DAVID J. WHEELER & STANLEY GILL, *Preparation of Programs for an Electronic Digital Computer*, Second Edition, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1957, xiv + 238 p., 23 cm. Price $7.50

The intended purpose of this book is stated on the cover sheet from which the following is selected: "Thoroughly revised and expanded to include machines other than the Electronic Delay Storage Automatic Computer (EDSAC), this Second Edition offers a general introduction to programming for any computer of the stored-program type. It is designed for those using electronic digital computers, for those putting new machines into operation, and for those wishing to assess the possible application of such computers to their own problems."

To justify the fact that most of the material is developed in terms of the EDSAC the authors state that: "Programming cannot be taught in the abstract nor can it

be learned without practice, and any book on programming must use the order code of some particular machine, real or hypothetical." They go on to say that the EDSAC lends itself well to this purpose, being a single address binary machine and therefore of a common type. Moreover, in basing instruction on the EDSAC there is a great deal of experience to draw upon.

The book is divided into three main parts.

Part One consists of eight chapters mainly devoted to the fundamentals of programming. Features of the EDSAC are gradually introduced as required for development. Many illustrative examples appear and are explained with a great deal of care. Exercises are included for which specimen solutions appear in Appendix 5.

One chapter is devoted to a comparative discussion of other types of storage-program digital computers. A chapter on subroutine libraries deals with their organization and use. The EDSAC library is described in detail with short accounts given of such topics as Quadrature, Chebyshev polynomials, etc. Another chapter is devoted to examples of complete programs for the EDSAC. The final chapter of Part I deals with the subject of automatic programming.

Part Two is entirely concerned with specifications for the EDSAC library subroutines. Here are to be found subroutines performing such functions as: arithmetic operations not contained in the order code; error diagnosis; generation of the elementary functions; integration of differential equations; numerical quadrature; input and output.

Part Three, consisting of 37 pages, contains the complete programs of subroutines selected from the EDSAC subroutine library.

Five appendices cover in connection with the EDSAC the input and output codes; order code and controls; the initial input routine; control combinations; and specimen solutions to the programming exercises.

The bibliography consists of 46 references on machines and programming and eleven references on numerical analysis.

It is the opinion of the reviewer that the book is clearly written and that the intentions of the authors are realized.

<div style="text-align: right">E. E. Osborne</div>

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183[Z].—C. C. Gotlieb & J. N. P. Hume, *High-Speed Data Processing*, McGraw-Hill, New York, 1958, xi + 238 p., 23 cm. Price $9.50.

The authors are associate professors of the University of Toronto, one in the Computation Centre and the other in the Department of Physics. Their interest in writing this book was to furnish study material for their students. There is a list of 52 problems, ranging from writing decimal numbers in binary notation to diagnosing errors in programs.

Despite this object and these problems the book is more a treatise than a text book. It attempts to present the computer and what it can do to the potential user. It describes and compares several designs, from the LGP-30 to the IBM 709. It emphasizes the intermediate machines, like the IBM 650, the File Computer,

the Datatron 205, and the Elecom 125. It emphasizes them to such an extent that a student might easily fail to appreciate the power or complexity of the bigger units.

The first 95 pages are devoted to describing the machines, how information is represented, and how instructions are executed. Various means of organizing and implementing the machine functions are described. In order to be specific the authors introduce a Hypothetical Machine for which examples in coding are given throughout the rest of the book.

There are fifty pages on the analysis, programming, and coding of problems. Only very general comments are made on analysis. Programming is considered at length. Loops, flow diagrams, linked subroutines (to which control is transferred and from which it jumps back), and scaling are described and illustrated by examples. The examples are mostly from business applications, payroll, amortization, taxes, inventory, and billing.

There are sixty pages on those properties of the machines which most concern the programmer and operator. One is reliability and checking, both built-in and programmed. With this is a treatment of the finding of programmer errors. Stepping slowly through the program, memory dumps, and interpretive routines are some of the techniques discussed. An interpretive routine for the Hypothetical Machine is diagrammed and coded to make crystal clear exactly what is meant. There is a discussion of looping, with and without B-boxes and other similar techniques, such as an auxiliary accumulator, with comparison of the times. Methods of saving time on a drum memory are discussed, although the claims for the efficacy of their techniques are very modest. The example of optimum programming leaves a false impression about the limitation on the number of accessible memory cells. A particular type of data is limited to twenty cells. The severity of this limitation is owing to the assignment of instruction addresses. Another technique of exploiting the cheapness of drum storage is multiple levels of memory. Magnetic files, their sorting and collating, maintenance and use, are given full treatment with examples.

There are sixty-eight pages on business applications. There is a chapter on insurance uses, including a comparison of costs with those of more traditional methods. There is another on accounting, including "inquiry systems", and banking operations. The chapter on control and prediction is interesting. A computer can be used to simulate the operation of a warehouse, and in this way give management some synthetic experience on which to base its planning. It can also schedule manufacturing in great detail, thus making economies possible. Control of inventory by computer has payed for itself in several industries. Some of the mathematics incident to these activities, such as linear programming, is presented with examples.

There is a chapter on automatic programming; compilers, interpreters, and some of the better known pseudo-codes.

On the whole this is a balanced book for its purpose. It presents to the student a panorama from the design of a computer through programming to use in advanced ways. No engineering knowledge is prerequisite. Care has been taken to balance one make of machine against another, so that the student will have some preparation no matter what installation he joins.

There is, in the reviewer's opinion, too much emphasis on the limited-capacity machine. The reader gets no idea of the impressive ability of the larger machines,

nor the slightest inkling of the exciting improvements of the near future. There is not even any critique of the shortcomings of the computers which are treated at length. The students are expected to grapple with the system and conquer it; "theirs not to reason why—." Perhaps this is a good attitude for budding "comptologists" to have.

In a few places the exposition is turgid. One of these is the justification of "rerun points" in a program. It is to be hoped that these can be clarified in a second edition.

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184[Z].—ALFRED K. SUSSKIND, Editor, *Notes on Analog-Digital Conversion Techniques*, The Technology Press, Massachusetts Inst. of Techn., Cambridge, Mass., 1957, viii + 400 p. Price $10.00

This book has bridged a big gap in the literary coverage of the electronic data processing field. There have been a considerable number of articles written about various analog-digital conversion devices and techniques but this is the first comprehensive text known to the reviewer. In the opinion of the reviewer, it is a very well written and comprehensive book about the field covered.

The book resulted from the notes used by Professor Susskind and others in teaching a course on analog-digital conversion techniques during the Special Summer Program at Massachusetts Institute of Technology.

The subject matter is divided into three parts. The first part pertains to systems aspects of digital information processing that influence the specifications for analog-to-digital and digital-to-analog conversion devices. In the second part, a detailed engineering analysis and evaluation of a variety of conversion devices is presented. The third part is devoted to a case study based on development work done at the Servomechanism Laboratory of the MIT Department of Electrical Engineering.

The inclusion of the third part makes the book particularly valuable to engineers in planning conversion schemes because it examines the features of many types of equipment presently available.

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TABLE ERRATA

267.—KULIK's, *Factor Table*, Carnegie Institute, 1948. Available on Microfilm in part, from the Carnegie Institute.

Errors in the Thirteenth Million of Kulik's Manuscript Factor Table

Kulik's monumental factor table has been frequently discussed in MTAC, for instance in Volume 2, Page 139 (July 1946) and Volume 3, Page 222 (July 1948). The latter note mentions that Carnegie Institution had available a microfilm of a part of Kulik's table, made from a photostatic copy secured through the efforts of D. N. Lehmer.

In March 1953, Palama and Poletti published a list of primes between 12,012,000 and 12,072,060 in the Boletino U.M.I. (MTAC Vol. 7.-p. 173, July 1953). Subsequently, Dr. N. G. W. H. Beeger published a tabulation of errors in Palama and Poletti's list (MTAC-Vol. 10-p. 54, Jan. 1956).

The writer has made a comparison of the entire list of primes of Palama and Poletti against the microfilm of Kulik's table, taking into account Dr. Beeger's corrections. The 25 errata in List A and List B were detected, but since the comparison was informal and not rigorous, a few errors may have been overlooked. There are a number of instances where marks have possibly been made by others on Kulik's manuscript, also places where he apparently erased mistakes incompletely. Because of the ambiguities that sometimes are thus created, it is possible that no two observers would reach agreement on exactly what constitute the true discrepancies between Kulik and Palama and Poletti. Reference to the actual table instead of the microfilm might assist an observer in reconciling certain situations. The 25 errata given are clearly errors; doubtful cases have been excluded.

All integers on list "A" have only the two prime factors shown.

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LIST A

Integers Shown as Prime by Kulik, But Which Are Not on Palama and Poletti's List of Primes

Integer	Prime Factors	
12,014,591	601	19,991
12,015,793	601	19,993
12,018,197	601	19,997
12,021,313	739	16,267
12,026,611	601	20,011
12,027,461	191	62,971
12,032,621	601	20,021
12,033,823	601	20,023
12,036,137	37	325,301
12,037,429	601	20,029
12,038,701	367	32,803
12,040,643	673	17,891
12,044,503	281	42,863
12,045,287	2953	4,079
12,048,247	601	20,047
12,050,651	601	20,051
12,057,863	601	20,063
12,066,113	1063	11,351

LIST B

Integers Shown as Composite by Kulik, But Shown as Prime in Palama and Poletti

12,042,761
12,044,147
12,045,181
12,045,227
12,047,569
12,066,163
12,071,779

NOTES

ANNOUNCEMENTS OF NEW AND NEARLY NEW JOURNALS AND SERIES

(1) *Chiffres: Revue de l'Association Française de Calcul*—published 3 times a year. The journal is the official publication of the new "association" which hopes to spread the new techniques of computation to as large an audience of research workers as possible. The first issue (March 1958) contains among its papers an article on the numerical solution of Poisson's equation, the expansion of π to ten thousand decimal places, several book reviews, the review of a new table of trigonometric functions. This journal promises to contribute valuably to the literature of numerical analysis. Subscriptions may be obtained by writing to M. Paul Rapin, 5, rue Général-Lanrezac, Neuilly (Seine), France.

(2) IBM, *Journal of Research and Development*—published quarterly by International Business Machines Corporation, 590 Madison Ave., New York 22, New York. Subscriptions are $3.50 per year in the United States and North America, $4.50 per year elsewhere. The journal is available for the quick publication and dissemination of the research activities of scientists and engineers of IBM. The IBM Corporation deserves much applause for initiating this scholarly activity. The range of subjects covered so far is very broad, e.g. from the construction of a heart-lung machine to the efficient calculation of arctan x on a digital computer. Most of the papers are in "some way" connected with research and development in the field of electronic computers. The journal is beautifully printed, with many excellent drawings and pictures, on slick paper (à la Life, Time, etc.).

(3) IRE, *Transactions on Electronic Computers*—published quarterly by the Institute of Radio Engineers, Inc., for the Professional Group on Electronic Computers. The articles printed are mainly on analogue and digital computers, their design and application. A complete reprinting of reviews of books in the fields of numerical analysis is included in the extensive section "Reviews of Current Literature". Subscription requests should be sent to the Institute of Radio Engineers, 1 East 79 Street, New York 21, New York.

(4) *Acta Polytechnica*—Applied Mathematics and Computing Machinery Series. This series is one of several published by ATV (Denmark's Academy of Sciences). The series consists of short papers or monographs, individually published and bound, which originate at Scandinavian research organizations. Subscriptions to a whole series or the purchase of single copies may be arranged with Acta Polytechnica Publishing Office, Box 5073, Stockholm 5, Sweden.

I. E.

VOLUME XII

AUTHOR INDEX

Papers and Technical Notes

[Short papers and notes are marked (N) in this index]

INDEX OF REVIEWS BY AUTHOR OF WORK REVIEWED